

Decay of correlations for billiards with flat points II: cusps effect

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Abstract

In this paper we continue to study billiards with flat points, by constructing a special family of dispersing billiards with cusps. All boundaries of the table have positive curvature except that the curvature vanishes at the vertex of cusps, i.e. the boundaries intersect at the flat point tangentially. We study the mixing rates of this one-parameter family of billiards parameterized by $\beta \in (2, \infty)$, and show that the correlation functions of the collision map decay polynomially with order $\mathcal{O}(n^{-\frac{1}{\beta-1}})$ as $n \rightarrow \infty$. In particular, this solves an open question raised by Chernov and Markarian in [10].

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1 Background and the main results.

Dispersing billiards introduced by Sinai are uniformly hyperbolic and have strong statistical properties. However, if the billiard table has cusps or flat points, then its hyperbolicity is nonuniform and statistical properties deteriorate.

Billiards with flat points were constructed and studied by Chernov and Zhang in [13, 25, 26]. It was proved that the mixing rates vary between $\mathcal{O}(1/n)$ and exponentially fast depending on the parameter $\beta \geq 2$, as $n \rightarrow \infty$. The main reason is that there exists one periodic trajectory between two flat points, which acts as a trap to slow down the mixing rates of nearby trajectories. In [25, 26], statistical properties of a semidispersing billiard with flat points on the convex boundary were investigated. The decay rates were proven to be dominated by the so-called “channel effect”, which is essential for semidispersing billiards, with the existence of a pair of parallel trajectories tangential to all convex boundaries.

The first rigorous analysis of correlations for dispersing billiards with cusps was given by Chernov and Markarian in [10], where they proved the rates of correlations (see the correlation function defined as in (1.2)) decay at $\mathcal{O}((\ln n)^2/n)$, as $n \rightarrow \infty$. The rates were improved to $\mathcal{O}(n^{-1})$ in [14]. This model was further investigated in [1, 2]. In [10], Chernov and Markarian also raised an open question: “It is interesting to let the curvature vanish at the vertex of the cusp, would this affect the rate of the decay of correlations?” Another interesting question to ask is that since all known billiards have decay rates at least of order $\mathcal{O}(n^{-1})$, are there any billiards with slower mixing rates?

To answer both questions, in this paper we investigate the model proposed by Chernov and Markarian in [10], and show that this family of billiards enjoys arbitrarily slower decay rates. More precisely, we first take Machtas three-arc table Q_1 as studied by Chernov and Markarian [10], with boundary consisting of 3 smooth curves Γ'_i , $i = 1 \cdots, 3$; and Q_1 has three cusps at the intersection points. Then we smoothly deform these curves at their end points, and denote the new curves as Γ_i ; such that they all have zero derivatives up to $\beta - 1$ order at end points, and the β -th order derivative is not zero, for $\beta \in (2, \infty)$. We let $Q = Q_\beta$ be bounded by $\Gamma_1, \Gamma_2, \Gamma_3$, for $\beta \in (2, \infty)$. Indeed according to our above assumption, if we choose a Cartesian coordinate system (s, z) with origin at any of these cusp point, denoted as P , with the horizontal s -axis being the tangent line to the boundary of the billiard table, then the billiard table satisfies the following three conditions:

(h1) We assume for some small $\varepsilon_0 > 0$, the pair of boundary adjacent to P can be repre-

sented in the ε_0 -neighborhood of the cusp P as:

$$(1.1) \quad z_1(s) = \beta^{-1}s^\beta, \quad z_2(s) = -\beta^{-1}s^\beta, \quad \forall s \in [0, \varepsilon_0]$$

(h2) We also assume that the tangent line of the table at the cusp P will hit the opposite boundary at a point, called D , perpendicularly.

We will investigate the statistical properties of the billiard system on Q_β . The billiard flow Φ^t is defined on the unit sphere bundle $Q \times \mathbf{S}^1$ and preserves the Liouville measure. This type of billiards can be viewed as a special type of semi-dispersing billiards, as its boundary contains points with zero curvature. Semi-dispersing billiards have been proven to enjoy strong ergodic properties: their continuous time dynamics and the billiard ball maps are both completely hyperbolic, ergodic, K-mixing and Bernoulli, see [5, 18, 20, 21, 22, 9] and the references therein. However, these systems have quite different statistical properties depending on the geometric properties of the billiard table. Figure 1. describes a billiard table with cusp at the flat point P for $\beta > 2$.

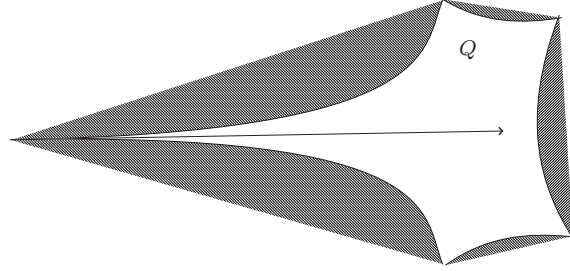


Fig. 1: A table with a cusp at a flat point for $\beta \in (2, \infty)$

There is a natural cross section \mathcal{M} in $Q \times \mathbf{S}^1$ that contains all postcollision vectors based at the boundary of the table ∂Q . The set $\mathcal{M} = \partial Q \times [-\pi/2, \pi/2]$ is called the collision space. Any postcollision vector $x \in \mathcal{M}$ can be represented by $x = (r, \varphi)$, where r is the arclength parameter along ∂Q , starting from an end point of ∂Q , measured in the clockwise direction; and $\varphi \in [-\pi/2, \pi/2]$ is the angle that x makes with the inward unit normal vector to the boundary.

The corresponding Poincaré map (or the billiard map) $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{M}$ generated by the collisions of the particle with ∂Q preserves a natural absolutely continuous measure μ on the collision space \mathcal{M} , such that

$$d\mu = \frac{1}{2|\partial Q|} \cos \varphi dr d\varphi.$$

For any square-integrable observable $f, g \in L^2_\mu(\mathcal{M})$, *correlations* of f and g are defined by

$$(1.2) \quad \mathcal{C}_n(f, g, \mathcal{F}, \mu) = \int_{\mathcal{M}} (f \circ \mathcal{F}^n) g \, d\mu - \int_{\mathcal{M}} f \, d\mu \int_{\mathcal{M}} g \, d\mu.$$

The mixing speed of the system $(\mathcal{M}, \mathcal{F}, \mu)$ is characterized by the *rate of decay of correlations*, i.e., by the speed of convergence to 0 of (1.2) for “good enough” functions f and g .

Let $S_{\pm n}$ be the singular set of the map $\mathcal{F}^{\pm n}$, for any $n \geq 1$, and $S_{n_1, n_2} := \cup_{m=n_1}^{n_2} S_m$ the union of these singular sets, for any integers n_1, n_2 . For any $\gamma \in (0, 1)$, let $\mathcal{H}(\gamma)$ be the set of all bounded real-valued functions $f \in L_\infty(\mathcal{M}, \mu)$, such that there exist integer n_1, n_2 , for any connected component $A \in \mathcal{M} \setminus S_{n_1, n_2}$, any $x, y \in A$,

$$(1.3) \quad |f(x) - f(y)| \leq \|f\|_\gamma \text{dist}(x, y)^\gamma,$$

with

$$\|f\|_\gamma := \sup_{A \in S_{n_1, n_2}} \sup_{x, y \in A} \frac{|f(x) - f(y)|}{\text{dist}(x, y)^\gamma} < \infty.$$

For every $f \in \mathcal{H}(\gamma)$ we define

$$(1.4) \quad \|f\|_{C^\gamma} := \|f\|_\infty + \|f\|_\gamma.$$

In this paper we obtain the following results.

Theorem 1. *For the family of billiards on Q_β defined as $((\mathbf{h1})\text{-(}\mathbf{h2}))$, with $\beta > 2$, Then for any $\gamma \in (0, 1]$, any observables $f, g \in \mathcal{H}(\gamma)$ on \mathcal{M} , there exists $C_{f, g} = C(f, g) > 0$, such that*

$$|\mu(f \circ \mathcal{F}^n \cdot g) - \mu(f)\mu(g)| \leq C_{f, g} n^{-\frac{1}{\beta-1}},$$

for $n \geq 1$.

For the case when $\beta = 2$, the system corresponds to the dispersing billiards with cusps and enjoys mixing rates of order $\mathcal{O}(n^{-1})$, see [10, 14].

Convention. We use the following notation: $A \sim B$ means that $C^{-1} \leq A/B \leq C$ for some constant $C > 1$. Also, $A = \mathcal{O}(B)$ means that $|A|/B < C$ for some constant $C > 0$. From now on, we will denote by $C > 0$ various constants (depending only on the table) whose exact values are not important.

2 General scheme

Based upon the methods by Young [24], a general scheme was developed by Markarian, Chernov and Zhang [19, 12, 13, 14, 15] on obtaining slow rates of hyperbolic systems with singularities and applied the method on different models. Let $M \subset \mathcal{M}$ be a nice subset, such that the induced map $F : M \rightarrow M$ is strongly (uniformly) hyperbolic. One can easily check that it preserves the measure μ_M obtained by conditioning μ on M . For our billiards, hyperbolicity deteriorates only as the moving particle gets trapped by a cusp, when it experiences a large number of rapid collisions near the corner point of the cusp.

In this paper, we first fix a number $K_0 > 1$, and call any sequence of successive collisions of length $> K_0$ in a cusp a *corner series*. In particular, we post an upper bound for K_0 , such that for any grazing collision $x \in \mathcal{M}$, its forward trajectory will enter a corner series of length $> K_0$. We thus define

$$(2.1) \quad M = \{x \in \mathcal{M} : \text{forward successive collisions of } x \text{ in any cusp has length} \leq K_0\}.$$

Clearly, there exists $\varphi_{K_0} \in (0, \pi/2)$, such that

$$(2.2) \quad |\varphi| \leq \varphi_{K_0}, \quad \forall \varphi \in M,$$

i.e. $M \subset \mathcal{M}$ stays away from $\varphi = \pm\pi/2$. For any $x \in M$ we call

$$R(x) = \min\{n \geq 1 : \mathcal{F}^n(x) \in M\}$$

the return time function and thus the return map $F : M \rightarrow M$ is defined by

$$(2.3) \quad F(x) = \mathcal{F}^{R(x)}(x), \quad \forall x \in M.$$

In order to prove Theorem 1, the strategy consists of two steps; they were fully described in [12, 14], and has been applied to several classes of billiards with slow mixing rates, see [13, 10], so we will not bring up unnecessary details here.

(F1) First, the map $F : M \rightarrow M$ enjoys exponential decay of correlations.

More precisely, for any Holder observables $f, g \in \mathcal{H}(\gamma)$ on M with Hölder exponent $\gamma \in (0, 1)$,

$$\left| \int_M (f \circ F^n) g d\mu_M - \int_M f d\mu_M \int_M g d\mu_M \right| \leq C \|f\|_{C^\gamma} \|g\|_{C^\gamma} \vartheta^n,$$

for some uniform constant $\vartheta = \vartheta(\gamma) \in (0, 1)$ and $C > 0$.

(F2) Second, the distribution of the return time function $R : M \rightarrow [1, \infty)$ satisfies:

$$\mu_M(R \geq n) \sim \frac{1}{n^{1+a}},$$

for some $a > 0$ and any large n .

It was proved in [12] that assumption **(F1)**-**(F2)** imply polynomial decay rates of order $\mathcal{O}(n^{-a}(\ln n)^{1+a})$. Also, the proof of **(F1)** is reduced in [12] to the verification of a one-step expansion condition, as well as the regularities of the invariant manifolds for the induced map. To improve the upper bound for the decay rates, one needs to analyze the statistical properties of the return time function. In [14], the upper bound of decay rates of correlations was improved by dropping the logarithmic factor. Mainly because points in the region $(R > n)$ has tendency to move to $(R < n)$ fast enough under further iterations of the billiard map.

The paper is organized as following. In Section 3, we investigate the asymptotic quantities for any typical, long corner series. In Section 4, we construct the induced system (F, M) , by removing those corner series. The hyperbolicity of the reduced map is also proved in Section 4. The assumption **(F2)** is verified in Section 5, by analyzing the distribution of the return time function. The exponential decay of correlations for the reduced map and **(F1)** was proved in Section 6, by verifying the One-step expansion estimates, see Lemma 8. In Section 7, we get the improved upper bound using method as in [14].

3 The corner series

In this section, we first investigate the geometry of corner series, which correspond to certain billiard trajectories entering the cusp and experiencing a large number of reflections there before getting out. To simplify our analysis we consider here the cusp made by Γ_1, Γ_1 with a common tangent line at the end point P . For any $N \geq K_0$, we define M_N to be the set of points in M whose forward trajectories go down the cusp at P for a corner series of length N . For simplicity, **we assume the flat point P has r -coordinate r_f .**

Let $N > K_0$ be the number of reflections in the corner series starting from a vector $x \in M_N$. For any $x = (r, \varphi) \in M_N$, let $x_N = Fx$, and $x_n := \mathcal{F}^n x = (r_n, \varphi_n)$, for $n = 1, \dots, N-1$, denote the set of all points of reflection on $\Gamma'_1 \cup \Gamma'_2$. We also call $\{x_n, n = 1, \dots, N-1\}$ a corner series of length N generated by $x \in M_N$.

We choose a Cartesian coordinate system (s, z) with origin at P , the horizontal s -axis being the tangent line to the boundary of the billiard table. By **(h1)**, for some small $\varepsilon_0 > 0$, the pair of boundary adjacent to P can be represented in the ε_0 -neighborhood of the cusp P as:

$$(3.1) \quad z(s) = \pm \beta^{-1} s^\beta, \quad \forall s \in [0, \varepsilon_0]$$

We denote s_n to be the s -coordinate of the base point of $x_n = (r_n, \varphi_n)$. By the smoothness of the boundary curves,

$$(3.2) \quad |r_n - r_f| = \int_0^{s_n} \sqrt{1 + |z'(s)|^2} ds = s_n + \mathcal{O}(s_n^\beta).$$

To estimate the tail distribution of $\mu_M(R \geq n)$, we will fix a large number N_0 , and only consider those corner series, such that $N > N_0$. We will also work with more convenient coordinates:

$$\gamma_n = \pi/2 - |\varphi_n|, \quad \text{and} \quad \alpha_n = \tan^{-1}(s_n^{\beta-1}).$$

Note that by (3.1), the tangent vector of ∂Q at s_n is $(1, s_n^{\beta-1})$, which implies that

$$(3.3) \quad \alpha_n = \tan^{-1}(s_n^{\beta-1}) = s_n^{\beta-1} + \mathcal{O}(s_n^{3(\beta-1)})$$

stands for the angle of the tangent vector at s_n made with the horizontal axis, or equivalently, with the tangent line through the flat point P . Note that both α_n and γ_n are positive for $1 \leq n \leq N-1$; α_n are all small if N is large enough. While γ_n are initially small, they slowly grow to about $\pi/2$ for $n \sim N/2$, and then again decrease and get small. We use notations similar to that of [10], and define

$$\alpha_{\bar{N}} := \min\{\alpha_n : 1 \leq n \leq N\}.$$

It was proved in [10] Lemma 3.1, that \bar{N} is almost the middle point of N , i.e. $|\bar{N} - N/2| \leq 2$, if $\beta = 2$. The proof only relies on the symmetry of the boundary of the billiard table near the cusp, so it also apply to our case words by words, even for $\beta > 2$. We further subdivide the corner series into three segments. We fix a small enough $\bar{\gamma}$ and let

$$N_1 = \max\{n \leq \bar{N} : \gamma_n < \bar{\gamma}\}, \quad N_3 = \max\{n \geq \bar{N} : \gamma_n > \bar{\gamma}\}.$$

And put $N_2 = \bar{N}$. We call the segment on $[1, N_1]$ the “entering period” in the corner series, the segment $[N_1 + 1; N_3 - 1]$ the “turning period”, and the segment $[N_3, N]$ its “exiting period”. Clearly $|N_3 + N_1 - N| \leq 2$.

By the symmetry of the billiard table, it is enough to consider the first half of the series, $1 \leq n \leq N_1$.

Using these relations, one has the following proposition for a corner series of length N generated by any $x \in M_N$.

Proposition 2. *The following are true:*

- (1) $N_1 \sim N_2 - N_1 \sim N_3 - N_2 \sim N - N_3 \sim N$. i.e. all three segments in the corner series have length of order N ;
- (2) $\alpha_1 \sim N^{-\frac{\beta}{2\beta-1}}$, $\alpha_n \sim n^{-1} \sim N^{-1}$, for $n \in [N_1, N_2]$;
- (3) $\alpha_n \sim (n^{\beta-1} N^\beta)^{-\frac{1}{2\beta-1}}$, for $n \in [1, N_1]$;
- (4) $\gamma_1 = \mathcal{O}(N^{-\frac{\beta}{2\beta-1}})$, $\gamma_2 \sim N^{-\frac{\beta}{2\beta-1}}$;
- (5) $\gamma_n \sim (nN^{-1})^{\frac{\beta}{2\beta-1}}$, for $n \in [1, N_1]$;
- (6) For N sufficiently large, the quantity $\{H_N((r_n, \varphi_n)), n = 1, \dots, N-1\}$ is almost invariant along a corner series of length N :

$$H_N((r_n, \varphi_n)) = |r_n - r_f|^\beta \cos \varphi_n = C_N + \mathcal{O}(N^{-\frac{2\beta-1}{\beta-1}}),$$

for any $n = N_1, \dots, N_2$, with $C_N = CN^{-\frac{\beta}{\beta-1}}$, for some uniform constant $C > 0$. For $n = 1, \dots, N_1$, we have

$$H_N((r_n, \varphi_n)) = |r_n - r_f|^\beta \cos \varphi_n = C_N + \mathcal{O}(n^{-1}N^{-\frac{\beta}{\beta-1}}).$$

Proof. By the symmetric property of Γ'_1 and Γ'_2 , we will now only concentrate on $n = 1, \dots, N_2 \sim N/2$. Note that both α_n and γ_n are positive for $1 \leq n \leq N_2$. $\{\alpha_n\}$ is a decreasing sequence, and are all small; while γ_n is an increasing sequence, which is initially small, the terms slowly grow to about $\pi/2$ for $n \sim N/2$.

The following equations are simple geometric facts:

$$(3.4) \quad \gamma_{n+1} = \gamma_n + \alpha_n + \alpha_{n+1},$$

$$(3.5) \quad s_{n+1} = s_n - \tau_n \cos(\gamma_n + \alpha_n),$$

where we denote

$$(3.6) \quad \tau_n = \frac{1}{\beta} \cdot \frac{s_n^\beta + s_{n+1}^\beta}{\sin(\gamma_n + \alpha_n)}.$$

Here τ_n is the free path between two collisions based at s_n and s_{n+1} .

Now we denote $v_n = \gamma_n + \alpha_n$, then

$$(3.7) \quad v_{n+1} = v_n + 2\alpha_{n+1}.$$

In addition (3.5) can also be written as

$$(3.8) \quad s_{n+1} = s_n - \frac{s_{n+1}^\beta + s_n^\beta}{\beta \tan v_n}.$$

Using the mean-value theorem, we know for any $d > 1$, and n large, there exists $s_{n,d,*} \in [s_{n+1}, s_n]$, such that

$$(3.9) \quad s_{n+1}^d - s_n^d = d s_{n,d,*}^{d-1} (s_{n+1} - s_n) = -d s_{n,d,*}^{d-1} \frac{s_{n+1}^\beta + s_n^\beta}{\beta \tan v_n}.$$

Combining with (3.3), we know that by the mean-value theorem, there exists $s_{n,*} \in [s_n, s_{n+1}]$, such that

$$\begin{aligned} \alpha_{n+1} - \alpha_n &= \tan^{-1}(s_{n+1}^{\beta-1}) - \tan^{-1}(s_n^{\beta-1}) = \frac{s_{n+1}^{\beta-1} - s_n^{\beta-1}}{1 + s_{n,*}^{2(\beta-1)}} \\ &= (s_{n+1}^{\beta-1} - s_n^{\beta-1})(1 - s_{n,*}^{2(\beta-1)} + \mathcal{O}(s_n^{4(\beta-1)})) \\ &= -(\beta - 1) s_{n,\beta-1,*}^{\beta-2} \frac{s_{n+1}^\beta + s_n^\beta}{\beta \tan v_n} (1 - s_n^{2(\beta-1)} + \mathcal{O}(s_n^{3(\beta-1)} / \tan v_n)) \end{aligned}$$

where we used (3.9) for $d = \beta - 1$ in the last step. Using (3.9) one more time, we get

$$\begin{aligned} s_{n+1}^\beta s_{n,\beta-1,*}^{\beta-2} + s_n^\beta s_{n,\beta-1,*}^{\beta-2} &= s_{n+1}^{2\beta-2} + s_n^{2\beta-2} + s_{n+1}^\beta (s_{n,\beta-1,*}^{\beta-2} - s_{n+1}^{\beta-2}) + s_n^\beta (s_{n,\beta-1,*}^{\beta-2} - s_n^{\beta-2}) \\ &= s_{n+1}^{2\beta-2} + s_n^{2\beta-2} + \mathcal{O}(s_n^{3(\beta-1)}/\tan v_n) \end{aligned}$$

Combining above facts, as well as (3.3), we get

$$(3.10) \quad \alpha_{n+1} - \alpha_n = -\bar{a} \cdot \frac{\alpha_{n+1}^2 + \alpha_n^2}{2 \tan v_n} + \mathcal{O}(\alpha_n^3 (\tan v_n)^{-2}) = \alpha_n - \bar{a} \cdot \frac{\alpha_n^2}{\tan v_n} + \mathcal{O}(\alpha_n^3 (\tan v_n)^{-2}),$$

where $\bar{a} = 2(\beta - 1)/\beta$.

By (3.7) and (3.8), we have

$$(3.11) \quad v_2 > 2\alpha_2, \quad \sum_{n=1}^{N_2} \alpha_n \leq v_{N_2}/2 \leq \pi/4.$$

(3.10) implies that

$$\bar{a} \cdot \frac{\alpha_n}{\tan v_n} - 1 = -\frac{\alpha_{n+1}}{\alpha_n} + \mathcal{O}(\alpha_n^2 (\tan v_n)^{-2}) < 1.$$

Thus

$$(3.12) \quad \frac{\alpha_n}{\tan v_n} < 1/\bar{a}.$$

We denote $A_n = s_n^\beta \sin v_n$, for $n = 1, \dots, N_2$. We would like to show that $A_n \sim A_{N_2}$ is almost invariant, as long as N is large. To see this, using (3.7)-(3.8), we get

$$\begin{aligned} e_n &:= A_{n+1} - A_n = s_{n+1}^\beta \sin v_{n+1} - s_n^\beta \sin v_n \\ &= (s_{n+1}^\beta - s_n^\beta) \sin v_{n+1} + s_n^\beta (\sin v_{n+1} - \sin v_n) \\ &= \beta s_{n,*}^{\beta-1} (s_{n+1} - s_n) \sin v_{n+1} + s_n^\beta \cos v_n^* (v_{n+1} - v_n) \\ &= -\frac{s_{n,*}^{\beta-1} (s_n^\beta + s_{n+1}^\beta) \sin v_{n+1}}{\tan v_n} + 2s_n^\beta s_{n+1}^{\beta-1} \cos v_n^* \\ &= -2s_n^{2\beta-1} \cos v_n + 2s_n^{2\beta-1} \cos v_n^* + \mathcal{O}(s_n^{3\beta-2} \cos v_n) \\ (3.13) \quad &= 4s_n^{3\beta-2} \sin v_n + \mathcal{O}(s_n^{3\beta-2} \cos v_n), \end{aligned}$$

where $s_{n,*} \in [s_{n+1}, s_n]$ and $v_n^* \in [v_n, v_{n+1}]$.

Note that for $n \in [N_1, N_2]$, $\bar{\gamma} < v_n \leq \pi/2$. This above estimation implies that

$$s_n^\beta \sin v_n = s_{N_2}^\beta \sin v_{N_2} + \mathcal{O}\left(\sum_{k=n}^{N_2} s_k^{3\beta-2}\right),$$

which implies that the error term satisfies

$$\sum_{k=n}^{N_2} s_k^{3\beta-2} \leq s_n^\beta.$$

Using (3.8), we know that

$$s_{n+1}^{2\beta-1} - s_n^{2\beta-1} = -\frac{2\beta-1}{\beta} \frac{s_{n+1}^{3\beta-2}}{\tan v_n} + \mathcal{O}\left(\frac{s_n^{4\beta-3}}{\tan v_n}\right).$$

If we sum over for $k \in [n, N_2]$, this implies that

$$s_n^{2\beta-1} = s_{N_2}^{2\beta-1} + \frac{2\beta-1}{\beta} \sum_{k=n}^{N_2} \frac{s_{k+1}^{3\beta-2}}{\tan v_k} + \mathcal{O}\left(\sum_{k=n}^{N_2} \frac{s_k^{4\beta-3}}{\tan v_k}\right).$$

Again using the fact that $v_n < \pi/2$, for $n \in [1, N_2]$, then we get

$$\sum_{k=n}^{N_2} s_{k+1}^{3\beta-2} \cos v_k \leq \frac{2\beta}{2\beta-1} s_n^{2\beta-1} < s_n^{2\beta-1}.$$

Next we fix another very small number $\bar{\gamma}_1$, and $\tilde{N}_2 < N_2$, such that $\tilde{N}_2 = \max\{n < N_2 : v_n < \pi/2 - \bar{\gamma}_1\}$. Then for any $n \in [1, \tilde{N}_2]$, then we get

$$\sin \bar{\gamma}_1 \sum_{k=n}^{\tilde{N}_2} s_{k+1}^{3\beta-2} \leq \sum_{k=n}^{\tilde{N}_2} s_{k+1}^{3\beta-2} \cos v_k < s_n^{2\beta-1}.$$

Combining the above estimations together with the expression for e_n , we know that

$$\sum_{k=n}^{\tilde{N}_2} |e_k| \leq 4s_n^{2\beta-1} / \sin \bar{\gamma}_1.$$

This suggests that for N large enough, for any $n = 1, \dots, \tilde{N}_2$, the quantity

$$s_n^\beta \sin v_n = s_{\tilde{N}_2}^\beta \sin v_{\tilde{N}_2} + \mathcal{O}(s_n^{2\beta-1}).$$

should be almost invariant. We define

$$(3.14) \quad C_N := s_{\tilde{N}_2}^\beta \sin v_{\tilde{N}_2} = s_{\tilde{N}_2}^\beta \cos \bar{\gamma}_1,$$

and

$$(3.15) \quad A_n := s_n^\beta \sin v_n.$$

Then we have shown

$$(3.16) \quad A_n = C_N(1 + \mathcal{O}(C_N^{-1}s_n^{2\beta-1})) \sim C_N,$$

for any $n \in [1, \tilde{N}_2]$.

Moreover for any $n \in [N_1, \tilde{N}_2]$, using the fact that $\bar{\gamma} < v_n < \pi/2 - \bar{\gamma}_1$, thus

$$(3.17) \quad s_n \sim s_{\tilde{N}_2}, \quad \forall n \in [N_1, \tilde{N}_2]$$

Below, we will only concentrate on $n \in [1, N_1]$. Note (3.16) implies that for $n \in [1, N_1]$, we have

$$(3.18) \quad s_n^\beta \sin v_n = C_N^{-1} + \mathcal{O}(s_n^{2\beta-1}), \quad \text{and} \quad \sin v_n = \cos \bar{\gamma}_1 \cdot \frac{s_{\tilde{N}_2}^\beta}{s_n^\beta} + \mathcal{O}(C_N s_n^{\beta-1}) \sim \frac{s_{\tilde{N}_2}^\beta}{s_n^\beta}.$$

We now define

$$(3.19) \quad u_n = \frac{\alpha_n}{\alpha_{n+1}}, \quad w_n = \frac{v_n}{\alpha_n}$$

It is important to find the asymptotic for w_n and u_n . First note that equation (3.7) yields

$$(3.20) \quad w_{n+1} = w_n u_n + 2 = w_n + 2 + w_n(u_n - 1).$$

Since $u_n - 1 \geq 0$ and $w_n \geq 2$ by (3.11), thus

$$(3.21) \quad w_n \geq 2n,$$

for any $1 \leq n \leq N_1$.

Thus for $n \in [1, N_1]$, we have

$$u_n - 1 = \frac{\alpha_n - \alpha_{n+1}}{\alpha_{n+1}} = \frac{\bar{a}(\alpha_{n+1} + u_n \alpha_n)}{2 \tan v_n} + \mathcal{O}(\alpha_n^2) = \mathcal{O}\left(\frac{\alpha_n}{\tan v_n}\right) < C,$$

for some uniform constant $C > 0$. Combining with (3.11), it implies that for $1 \leq n \leq N_1$, $0 < u_n - 1 < C$ is uniformly bounded.

Moreover, (3.8) implies that

$$(3.22) \quad 1 - u_n^{-1} = \frac{v_n}{\tan v_n} \cdot \frac{\bar{a}(u_n^2 + 1)}{2w_n} + \mathcal{O}(\alpha_n^2) \leq \frac{\bar{a}}{w_n} + \mathcal{O}(\alpha_n^2) \leq \frac{\bar{a}}{2n} + \mathcal{O}(\alpha_n^2),$$

and

$$\alpha_{n+1} \geq \frac{\alpha_n}{1 + w_n^{-1}} + \mathcal{O}(\alpha_n^3) \geq \alpha_n \left(1 - \frac{1}{2n}\right) + \mathcal{O}(\alpha_n^3) + \mathcal{O}(\alpha_n n^{-2}).$$

This implies that

$$\alpha_n \geq \frac{\alpha_1}{2} \prod_{m=2}^n \left(1 - \frac{1}{m}\right) \geq \frac{\alpha_1}{cn}$$

for some constant $c > 0$.

We denote $u_n = 1 + b_n$, with $b_n \leq \frac{\bar{a}}{n}$, then one can check that

$$u_n^{-1} + u_n = 2 + \frac{b_n^2}{u_n}.$$

(3.22) implies that

$$u_n - 1 = \frac{v_n}{\tan v_n} \cdot \frac{\bar{a}(u_n^{-1} + u_n)}{2w_n} + \mathcal{O}(\alpha_n^2) = \frac{v_n}{\tan v_n} \cdot \frac{\bar{a}}{w_n} + \mathcal{O}(\alpha_n^2).$$

Now (3.20) implies that

$$w_{n+1} = w_n u_n + 2 \leq 2 + w_n(1 + b_n).$$

Combining (3.22) with (3.20) gives

$$(3.23) \quad w_{n+1} = w_n + 2 + \bar{a} + \mathcal{O}(n\alpha_n^2) + \mathcal{O}(n^{-2}) + \mathcal{O}(v_n^2).$$

By (3.18), we know that

$$\sin v_n \sim \frac{\alpha_{N_1}^{\frac{\beta}{\beta-1}}}{\alpha_n^{\frac{\beta}{\beta-1}}} = \prod_{k=n}^{N_1} u_k^{-\frac{\beta}{\beta-1}} \sim (n/N_1)^{\frac{\beta}{\beta-1}}.$$

Thus

$$v_n \sim (n/N_1)^2.$$

Let $\Gamma_n = \sum_{i=1}^n v_i^2$, one can now show that $\sum_{i=1}^n \Gamma_i/i^2$ is bounded:

$$\sum_{i=1}^n \Gamma_i/i^2 \leq C \sum_{i=1}^n i^3/N_1^4 \leq C,$$

for any $n \in [1, N_1]$. The lower bound in (3.21) now implies

$$(3.24) \quad w_n = w_1 + (2 + \bar{a})n + E_n,$$

where $E_n \leq C_1 \ln n + C_2 \Gamma_n$, for some constant $C_1, C_2 > 0$.

Now we use (3.24) to estimate u_n . Note that

$$\begin{aligned}
1 - u_n^{-1} &= \frac{v_n}{\tan v_n} \cdot \frac{\bar{a}(u_n^2 + 1)}{2w_n} + \mathcal{O}(\alpha_n^2) \\
&= \frac{\bar{a}}{(2 + \bar{a})n} \cdot \frac{v_n}{\tan v_n} \cdot \frac{1}{1 + \frac{\bar{a}(w_1 + C_1 \ln n + C_2 \Gamma_n)}{3n}} + \mathcal{O}(\alpha_n^2) \\
(3.25) \quad &= \frac{\bar{a}}{(2 + \bar{a})n + E_{2,n}} + \mathcal{O}(\alpha_n^2),
\end{aligned}$$

where $0 \leq E_{2,n} < C_3 \ln n + C_4 \Gamma_n$ for some uniform constants $C_i > 0$, $i = 1, 2, 3$. Using the definition of u_n , we have obtain

$$1 - u_n^{-1} = \frac{\alpha_{n+1} - \alpha_n}{\alpha_n} = \frac{\bar{a}}{(2 + \bar{a})n} + \mathcal{O}\left(\frac{1}{n^2 \ln n}\right).$$

Combining with (3.10), we get

$$(3.26) \quad \frac{\alpha_n}{\tan \gamma_n} = \frac{1}{(2 + \bar{a})n} + \mathcal{O}\left(\frac{1}{n^2 \ln n}\right).$$

This implies that

$$\frac{\bar{a}}{(2 + \bar{a})n} + \mathcal{O}(\alpha_n^2) \leq u_n - 1 \leq \frac{\bar{a}}{(2 + \bar{a})n + C_3 \ln n + C_4 \Gamma_n} + \mathcal{O}(\alpha_n^2).$$

Using the definition $\alpha_{n+1} = \alpha_n / u_n$, we get

$$\alpha_n = \alpha_1 \exp \left(- \sum_{i=1}^n \ln \left(1 + \frac{\bar{a}}{(2 + \bar{a})i + E_{2,i}} \right) \right) \sim \alpha_1 n^{-\bar{a}/(2+\bar{a})}.$$

Thus implies that for $n \in [1, N_1]$,

$$(3.27) \quad \alpha_n \sim \alpha_1 n^{-\bar{a}/(2+\bar{a})} \sim \alpha_1 n^{-\frac{\beta-1}{2\beta-1}}.$$

Combining with (3.24), we have

$$w_n = (w_1 + (2 + \bar{a})n)\alpha_n \sim (2 + \bar{a})\alpha_1 n^{\frac{2}{2+\bar{a}}} \sim \alpha_1 n^{\frac{\beta}{2\beta-1}}.$$

Since $v_n \sim \bar{\gamma}$, for $n \in [N_1, N_2]$, so

$$\alpha_1 \sim N_2^{-\frac{2}{2+\bar{a}}} \sim N_1^{-\frac{2}{2+\bar{a}}} \sim N^{-\frac{\beta}{2\beta-1}}.$$

Thus we have $N_1 \sim N_2 \sim N$.

This further implies that for $n \in [2, N_2]$, we have

$$(3.28) \quad \alpha_n \sim \frac{1}{(n^{\beta-1} N^\beta)^{\frac{1}{2\beta-1}}}, \quad \gamma_n \sim v_n \sim (nN^{-1})^{\frac{\beta}{2\beta-1}}.$$

In particular, for $n \in [N_1, N_2]$, using the above fact that $N_1 \sim N_2 \sim N$, we have

$$(3.29) \quad \alpha_n \sim n^{-1} \sim N^{-1}, \quad v_n \sim \bar{\gamma}.$$

Now we move back to improve the estimation of A_n . Note that

$$C_N = \sin v_{\tilde{N}_2} s_{\tilde{N}_2}^\beta = \cos \bar{\gamma}_1 s_{\tilde{N}_2}^\beta \sim N^{-\frac{\beta}{\beta-1}}.$$

Now by (3.16), we know that

$$A_n = s_n^\beta \sin v_n = C_N + \mathcal{O}(s_n^{2\beta-1}),$$

for any $n = 1, \dots, \tilde{N}_2$.

On the other hand, (3.13) implies that for $n = 1, \dots, N_2$,

$$A_n - A_{n+1} = \mathcal{O}(s_n^{3\beta-2} \sin v_n) + \mathcal{O}(s_n^{3\beta-2} \cos v_n).$$

Thus for $n \in [1, N_1]$, we have

$$A_n = A_{N_1} + \mathcal{O}\left(\sum_{k=n}^{N_1} s_k^{3\beta-2} \sin v_k\right) + \mathcal{O}\left(\sum_{k=n}^{N_1} s_k^{3\beta-2} \cos v_k\right) = A_N + \mathcal{O}(n^{-1} N^{-\frac{\beta}{\beta-1}}),$$

where we have used (3.28) in the last step. Moreover, using (3.29), we also get

$$A_n = s_n^\beta \sin v_n = C_N + \mathcal{O}(N^{-\frac{2\beta-1}{\beta-1}}),$$

for any $n = N_1, \dots, N_2$. Since $v_n = \gamma_n + \alpha_n$, we will now define a new quantity:

$$(3.30) \quad H_N((r_n, \varphi_n)) = |r_n - r_f|^\beta \cos \varphi_n = s_n^\beta \sin \gamma_n + \mathcal{O}(s_n^{2\beta} \sin \gamma_n).$$

The above estimation implies that

$$\begin{aligned} H_N((r_n, \varphi_n)) &= A_n - s_n^\beta (\sin \gamma_n - \sin v_n) \\ &= A_n + 2s_n^\beta \sin \frac{\alpha_n}{2} \cos(\gamma_n + \frac{\alpha_n}{2}) \\ &= A_n + \mathcal{O}(s_n^{2\beta-1}) = A_n + \mathcal{O}(n^{-1} N^{-\frac{\beta}{\beta-1}}) \\ &= C_N + \mathcal{O}(n^{-1} N^{-\frac{\beta}{\beta-1}}). \end{aligned}$$

For $n \in [1, N_1]$, we denote \mathcal{K}_n as the curvature of the boundary at x_n . Then one can check that by (1.1), the curvature satisfies

$$(3.31) \quad \mathcal{K}_n = (\beta - 1)s_n^{\beta-2} + \mathcal{O}(s_n^{\beta-1})$$

Using (3.6), we get

$$(3.32) \quad \begin{aligned} \frac{2\mathcal{K}_n\tau_n}{\sin \gamma_n} &= \frac{2(\beta - 1)(s_n^{2\beta-2} + s_n^{\beta-2}s_{n+1}^\beta)}{\beta \sin \gamma_n \sin v_n} \\ &= \frac{2(\beta - 1)(s_n^{2\beta-2} + s_n^{2\beta-2} + s_n^{\beta-2}(s_{n+1}^\beta - s_n^\beta))}{\beta \sin \gamma_n \sin v_n} \\ &= \frac{\bar{a}(\alpha_n^2 + \alpha_{n+1}^2)}{\sin v_n \sin \gamma_n} + \mathcal{O}(\alpha_n^3) \\ &= \frac{\bar{a}(1 + u_n^{-2})}{w_n^2} + \mathcal{O}(\alpha_n^3) \\ &= \frac{2\bar{a}}{(2 + \bar{a})^2 n^2} + \mathcal{O}(\alpha_n^3) \end{aligned}$$

where we have used $\bar{a} = 2(\beta - 1)/\beta$, as well as (3.8) and (3.9).

Using (3.6), we know that

$$\frac{\tau_{n+1}}{\tau_n} = \frac{s_{n+1}^\beta + s_{n+2}^\beta}{s_n^\beta + s_{n+1}^\beta} \cdot \frac{\sin(\gamma_n + \alpha_n)}{\sin(\gamma_{n+1} + \alpha_{n+1})}.$$

Note that for $n \in [1, N_1]$,

$$E_1 := \frac{s_{n+1}^\beta + s_{n+2}^\beta}{s_n^\beta + s_{n+1}^\beta} = \frac{u_{n+1}^{-\frac{\beta}{\beta-1}} + 1}{1 + u_n^{\frac{\beta}{\beta-1}}}.$$

Thus we have

$$E_1 = 1 - \frac{\beta}{(2\beta - 1)n} + \mathcal{O}(\alpha_n^2).$$

Moreover,

$$E_2 := \frac{\sin(\gamma_n + \alpha_n)}{\sin(\gamma_{n+1} + \alpha_{n+1})} = \frac{n\alpha_n}{(n+1)\alpha_{n+1}} + o(n^{-1}) = 1 - \frac{2}{(2 + \bar{a})n} + \mathcal{O}(\alpha_n^2).$$

Thus we have for $n \in [1, N_1]$,

$$(3.33) \quad \frac{\tau_{n+1}}{\tau_n} = E_1 \cdot E_2 = 1 - \frac{4}{(2 + \bar{a})n} + \mathcal{O}(\alpha_n^2) = 1 - \frac{2\beta}{(2\beta - 1)n} + \mathcal{O}(\alpha_n^2).$$

Combining with items (1)-(5), we get for $1 < n < N_1$,

$$\tau_n \sim \frac{1}{n^{\frac{2\beta}{2\beta-1}} N^{\frac{\beta}{(2\beta-1)(\beta-1)}}}.$$

For $N_1 < n < N_3$, we have $\alpha_n \sim N^{-1}$ and $\gamma_n > \bar{\gamma}$, which implies that

$$\tau_n \sim s_n \alpha_n \sim N^{-\frac{\beta}{\beta-1}}.$$

Thus the trajectory during each period in the corner series has the order of

$$\sum_{n=1}^{N_1} \tau_n \sim N^{-\frac{1}{\beta-1}}, \quad \sum_{n=N_1}^{N_2} \tau_n \sim N^{-\frac{1}{\beta-1}}.$$

□

Due to the time reversibility of billiard dynamics, all the asymptotic formulas obtained for the entering period remain valid for the exiting period.

4 Hyperbolicity of (F, M)

4.1 Stable/unstable cones

We let M be the collision space defined as in (2.1), and we define the return time function $R : M \rightarrow \mathbb{N}$ as well as the induced map $F : M \rightarrow M$ as (2.3). Then F preserves $\mu_M = \mu/\mu(M)$. Clearly, the induced map is indeed a dispersing billiard system. Thus using techniques in Chapter 4 of [9], one can obtain that the system (F, M) is uniformly hyperbolic.

In this section we investigate the expansion factors for vectors in unstable cones of the induced system (F, M, μ_M) . We denote $\mathcal{K}(x)$ as the curvature of the boundary at the base point of x .

We first recall that the differential of the billiard map \mathcal{F} satisfies:

$$(4.1) \quad D_x \mathcal{F} = \frac{-1}{\cos \varphi_1} \begin{pmatrix} \tau \mathcal{K}(r) + \cos \varphi & \tau \\ \tau \mathcal{K}(r) \mathcal{K}(r_1) + \mathcal{K}(r) \cos \varphi_1 + \mathcal{K}(r_1) \cos \varphi & \tau \mathcal{K}(r_1) + \cos \varphi_1 \end{pmatrix}.$$

Next we introduce the concept of the wave front. Let $V \in \mathcal{T}_x M$ be a tangent vector. For $\varepsilon > 0$ small, let us consider an infinitesimal curve $\gamma = \gamma(s) \subset M$, where $s \in (-\varepsilon, \varepsilon)$ is a parameter, such that $\gamma(0) = x$ and $\frac{d}{ds} \gamma(0) = V$. The forward (backward) trajectories of the points $y \in \gamma$, after leaving M , make a bundle of directed lines in Q , which is called the forward (backward) “wave front”. Let $\mathcal{B} = \mathcal{B}(x)$ ($\mathcal{B}^- = \mathcal{B}^-(x)$) be the curvature of the

orthogonal cross-section of the forward (backward) wave front at the point x with respect to the vector V . Indeed we have

$$(4.2) \quad \mathcal{B}^-(x_1) = \frac{\mathcal{B}(x)}{1 + \tau(x)\mathcal{B}(x)} = \frac{1}{\tau(x) + \frac{1}{\mathcal{B}(x)}} \quad \text{and} \quad \mathcal{B}(x) = \mathcal{B}^-(x) + \frac{2\mathcal{K}(x)}{\cos \varphi}.$$

where $x_1 = \mathcal{F}x$.

By our assumption on the table and the definition of M , there exist $\tau_{\min} > 0$, $0 < \mathcal{K}_{\min} < \mathcal{K}_{\max} < \infty$ such that for any $x \in M$,

$$(4.3) \quad \tau(x) > \tau_{\min}, \quad \mathcal{K}_{\min} \leq \mathcal{K}(x) \leq \mathcal{K}_{\max}.$$

Denote by $\mathcal{V} = d\varphi/dr$ the slope of the tangent line of W at x . Then \mathcal{V} satisfies

$$(4.4) \quad \mathcal{V} = \mathcal{B}^- \cos \varphi + \mathcal{K}(r) = \mathcal{B} \cos \varphi - \mathcal{K}(r).$$

We use the cone method developed by Wojtkowski [23] for establishing hyperbolicity for phase points $x = (r, \varphi) \in M$. In particular we study stable and unstable wave front. The relations in (4.2) imply that a dispersing wave front remains bounded away from zero.

Definition 3. *The unstable cone \mathcal{C}_x^u contains all tangent vectors based at x whose images generate dispersing wave fronts:*

$$\mathcal{C}_x^u = \{(dr, d\varphi) \in \mathcal{T}_x M : \mathcal{K}(r) \leq d\varphi/dr \leq \mathcal{K}(r) + \tau_{\min}^{-1}\}.$$

Similarly the stable cones are defined as

$$\mathcal{C}_x^s = \{(dr, d\varphi) \in \mathcal{T}_x M : -\mathcal{K}(r) \geq d\varphi/dr \geq -\mathcal{K}(r) - \tau_{\min}^{-1}\}.$$

We say that a smooth curve $W \subset M$ is an unstable (stable) curve for the system $(\mathcal{F}, \mathcal{M})$ if at every point $x \in W$ the tangent line $\mathcal{T}_x W$ belongs in the unstable (stable) cone \mathcal{C}_x^u (\mathcal{C}_x^s). Furthermore, a curve $W \subset M$ is an unstable (resp. stable) manifold for the system $(\mathcal{F}, \mathcal{M})$ if $\mathcal{F}^{-n}(W)$ is an unstable (resp. stable) curve for all $n \geq 0$ (resp. ≤ 0).

We consider a short unstable curve $W \subset M_N$ with equation $\varphi = \varphi(r)$, for some $N \geq 1$. Let $x = (r, \varphi) \in W$ and $V = (dr, d\varphi)$ be a tangent vector at x of W . Combining with (4.1) and (2.2), we know for any $x \in M$, the slope of the tangent vector of $\mathcal{F}W$ at $\mathcal{F}x$ satisfies:

$$(4.5) \quad \mathcal{K}(\mathcal{F}x) + \frac{\cos \varphi_{K_0}}{\tau_{\max} + \frac{1}{2K_{\min}}} \leq \mathcal{V}(\mathcal{F}x) = \mathcal{K}(\mathcal{F}x) + \frac{\cos \varphi(\mathcal{F}x)}{\tau(x) + \frac{\cos \varphi}{\mathcal{K}(x) + \mathcal{V}(x)}} \leq \mathcal{K}(\mathcal{F}x) + \frac{1}{\tau_{\min} + \frac{\cos \varphi_{K_0}}{2\mathcal{K}_{\max} + \tau_{\min}^{-1}}}.$$

This implies that for any $x \in M$, the unstable cone \mathcal{C}_x^u is already strictly invariant under \mathcal{F} . Thus we get $DF\mathcal{C}_{\mathcal{F}x}^u \subset \mathcal{C}_{\mathcal{F}x}^u$. Similarly one can check that $DF(\mathcal{C}_x^s) \supset \mathcal{C}_{\mathcal{F}x}^s$.

Next we will show that any unstable vector in \mathcal{C}^u gets uniformly expanded under DF . We introduce two metrics on the tangent space $\mathcal{T}M$. Let $x \in M_N$, for any $N \geq 1$.

(I) The first one is the so-called *p-metric* on vectors $dx = (dr, d\varphi)$ by

$$(4.6) \quad |dx|_p = \cos \varphi |dr|.$$

Put $\mathcal{F}x = (r', \varphi')$ and $dx' = (dr', d\varphi') = D\mathcal{F}(dx)$, for $n \geq 1$. The expansion factor is

$$(4.7) \quad \frac{|D\mathcal{F}(dx)|_p}{|dx|_p} = 1 + \tau(x)\mathcal{B}(x) \geq 1 + \frac{\tau(x)\mathcal{K}(r)}{\cos \varphi}.$$

(II) Now consider the expansion factor in the *Euclidean metric* $|dx|^2 = (dr)^2 + (d\varphi)^2$.

Note that for any $x = (r, \varphi) \in M_N$, with N large enough, $\cos \varphi$ is approximately 1 by our assumptions on the billiard table. Moreover, for any $dx = (dr, d\varphi) \in \mathcal{C}_x^u$, we have $d\varphi/dr$ is approximately 1. Thus for any vector $dx \in \mathcal{C}_x^u$,

$$(4.8) \quad \frac{|D\mathcal{F}(dx)|}{|dx|} = \frac{|D\mathcal{F}(dx)|_p}{|dx|_p} \frac{\cos \varphi}{\cos \varphi'} \frac{\sqrt{1 + (\frac{d\varphi'}{dr'})^2}}{\sqrt{1 + (\frac{d\varphi}{dr})^2}}.$$

Hence, the p -norm and the Euclidian norm are equivalent for tangent vectors at x and $\mathcal{F}x$. We let $x \in M_N$ and define $x_n = (s_n, \varphi_n) = \mathcal{F}^n x$, for $n = 1, \dots, N$, corresponding to iterations for a corner series of length N as introduced in the above section.

Proposition 4. *For any $N > K_0$, any unstable curve $W \subset M_N$ such that $F^{-1}W$ is also unstable, let $x = (r, \varphi) \in W$. The total expansion factor for unstable vectors in the course of the corner series of N collisions has lower bound*

$$\Lambda(x) := \frac{|DF(dx)|}{|dx|} \geq CN^{1 + \frac{\beta}{(2\beta-1)(\beta-1)}},$$

where $C > 0$ is a constant. Its precise asymptotic is

$$\Lambda(x) \sim N^{1 + \frac{\beta}{(2\beta-1)(\beta-1)}} \left(1 + \frac{N^{-\frac{\beta}{2\beta-1}}}{\cos \varphi_1} \right) \left(1 + \frac{N^{-\frac{\beta(\beta-2)}{(2\beta-1)(\beta-1)}}}{\cos \varphi_N} \right).$$

The proof of this proposition is rather lengthy, so we put it in the appendix.

5 Distribution of the return time function

In this section, we use the results of the previous sections to analyze the distribution of the return time function R , together with its level set M_n , $n \geq K_0$, which consists of points

whose trajectories go down a cusp and experience there a corner series of exactly $n - 1$ collisions. We will use standard facts of the theory of dispersing billiards [3, 4, 6, 9, 10]. For example, the domains M_n are bounded by singularity curves of the map F . These singular curves are made of unstable curves and the preimage of ∂M . Due to the time-reversibility of the billiard dynamics, FM_n is obtained by reflecting M_n across the line $\varphi = 0$. Moreover, a point (r, φ) is a singularity point for the map F (i.e. $F(r, \varphi)$ or its differential is not well-defined) if and only if $(r, -\varphi)$ is a singularity point for its inverse F^{-1} (i.e. $F^{-1}(r, -\varphi)$ or its differential is not well-defined).

By our assumption, we know the singular trajectory running out of the cusp at P will land on the point D on the opposite side Γ_3 perpendicularly. Let $x_D = (r_D, 0)$, where r_D is the r -coordinate of the singular point x_D . Indeed x_D belongs to a singular curve, which we call s_0 , that is made of all grazing collisions on ∂Q . One can check using (4.1) that the slope of the tangent vector at $x = (r, \varphi) \in s_0$ satisfies

$$d\varphi/dr = -(K(x) + \cos \varphi / \tau(x)).$$

Thus in the vicinity of x_D , the curve s_0 can be approximated by a line with slope $-K(x_D) - l_D^{-1}$, where l_D is the distance between the cusp P and the base point of x_D in the billiard table.

The singular curves of F near x_D consist of two symmetric sequences of singularity curves $\{s'_n\}$ and $\{s''_n\}$, approaching x_D from both sides of s_0 . More precisely, s'_n consists of points whose trajectories enter the cusp by hitting Γ_1 first, and the last collision in the corner series is grazing. Similarly, s''_n hits Γ_2 first, where the last collision is grazing when exiting the cusp. Denote by M'_n the strip bounded between s'_n, s'_{n+1} ; and M''_n bounded by s''_n and s''_{n+1} . Then $M_n = M'_n \cup M''_n$. By the symmetric property, it is enough to concentrate on M'_n .

Fig 2. shows the structure of the singular curves near x_D . In order to determine the rates of the decay of correlations we need certain quantitative estimates on the measure of the regions M_n , or $\{R \geq n\}$. Note that for any $N \geq K_0$, the forward images $\{\mathcal{F}^k M_N, k = 1, \dots, N - 1\}$ fill entirely the regions squeezed between the two curves H_N and H_{N+1} . By Proposition 2, we know that the set $\cup_{n=1}^{N-1} \mathcal{F}^n M'_N \subset \mathcal{M}$ is bounded by the line $r = r_f, \varphi = \pi/2$ and a curve described implicitly by the equation of H_N :

$$(5.1) \quad r^\beta = \frac{C_N}{\sin \varphi} (1 + \mathcal{O}(r^{2\beta-1} C_N^{-1}))$$

Equivalently H_N has equation given by:

$$r = \frac{C_N^{\frac{1}{\beta}}}{(\sin \varphi)^{\frac{1}{\beta}}} + \mathcal{O} \left(\frac{r^{2\beta-1}}{C_N^{1-\frac{1}{\beta}} \sin^{\frac{1}{\beta}} \varphi} \right)$$

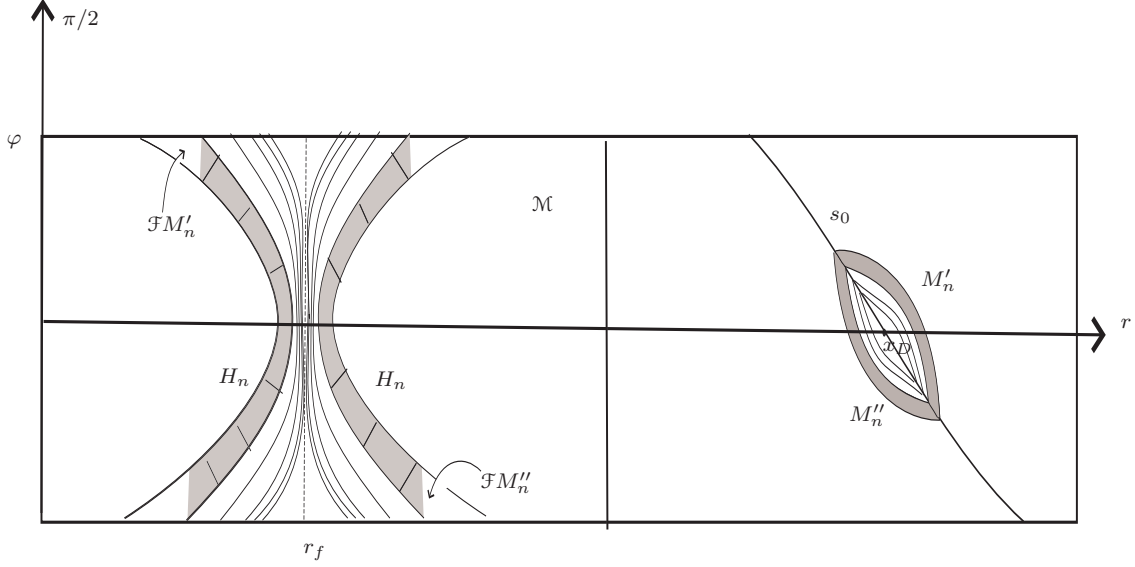


Figure. 2: Singularity curves of F in the vicinity of x_D ; and their forward images under \mathcal{F} near the cusp in \mathcal{M} (that are bounded by H_n).

We extend the definition of R from M to \mathcal{M} , such that for any $x \in \mathcal{M}$,

$$R(x) = \min\{n \geq 1 : \mathcal{F}^n x \in M\}.$$

Using Propostion 2, we have for $N > K_0$,

$$\begin{aligned} \mu(x \in \mathcal{M} : R \geq N) &= \sum_{m \geq N} \sum_{k=0}^{m-N} \mu(\mathcal{F}^k M_m) \sim \sum_{m \geq N} \sum_{k=0}^{m-1} \mu(\mathcal{F}^k M_m) \\ (5.2) \quad &\sim \int_0^\pi \frac{C_N^{\frac{1}{\beta}}}{\sqrt[\beta]{\sin \varphi}} \sin \varphi d\varphi = C_N^{\frac{1}{\beta}} \int_0^\pi (\sin \varphi)^{1-\frac{1}{\beta}} d\varphi \sim N^{-\frac{1}{\beta-1}}. \end{aligned}$$

Another method of calculation of the measure of M_N relies on the factor of expansion of unstable manifolds $W \subset M_N$ under the map F . We use Proposition 4 to estimate M_N .

Lemma 5. *For any $N \geq 1$, M_N has measure $\sim N^{-2-\frac{1}{\beta-1}}$. Thus $\mu_M(x \in M : R \geq N) \sim N^{-1-\frac{1}{\beta-1}}$; and $\mu(x \in \mathcal{M} : R \geq N) \sim N^{-\frac{1}{\beta-1}}$.*

Proof. To determine the dimensions of the strips M'_N , observe that the two intersection points in s'_N with the curve s_0 are located farthest from the central point x_D , which are made by trajectories whose very first collision in the cusp is grazing. Let x be such an end point, and $x_1 = \mathcal{F}x$. By our assumption, we know that the tangent vector at $x_1 = (r_1, \varphi_1)$

makes an angle approximately $r_1^{\beta-1}$ with the horizontal line. By Proposition 2, we know that $\alpha_1 \sim N^{-\frac{\beta}{2\beta-1}}$. Thus we conclude that the trajectory originates at the distance $\sim N^{-\frac{\beta}{2\beta-1}}$ from the point x_D . Thus the diameter of M_N (i.e. the ‘length’ of these strips) is $\sim N^{-\frac{\beta}{2\beta-1}}$.

Due to the time-reversibility of the billiard dynamics, the singular curves in FM have a similar structure. Furthermore, the short sides of M'_N stretch completely under FM'_N , and are transformed into long sides of FM'_N . Let $W \subset M'_N$ be a short unstable curve that stretches completely in M'_N between two long sides. Then $|FW| \sim N^{-\frac{\beta}{2\beta-1}}$. Thus by Proposition 4, the expansion factor on W is $\sim N^{1+\frac{\beta}{(2\beta-1)(\beta-1)}}$. This implies that the width is:

$$(5.3) \quad |W| \sim N^{-\frac{\beta}{2\beta-1}} / N^{1+\frac{\beta}{(2\beta-1)(\beta-1)}} \sim N^{-1-\frac{\beta}{2\beta-1}-\frac{\beta}{(2\beta-1)(\beta-1)}}.$$

Note that the length of M'_N is $\sim N^{-\frac{\beta}{2\beta-1}}$, and the density on M'_N is $\sim \cos \varphi \sim 1$, as the collision at x is almost perpendicular. According to (5.3), the measure of M_N is of order

$$\mu_M(M_N) \sim N^{-1-\frac{\beta}{(2\beta-1)(\beta-1)}} \cdot N^{-\frac{2\beta}{2\beta-1}} = N^{-2-\frac{1}{\beta-1}}.$$

□

It is immediate that for any sufficiently large N ,

$$(5.4) \quad \mu_M(x \in M : R(x) \geq N) \sim N^{-1-\frac{1}{\beta-1}} \quad \text{and} \quad \mu(x \in \mathcal{M} : R(x) \geq N) \sim N^{-\frac{1}{\beta-1}}.$$

This verifies condition **(F2)** with $a = \frac{1}{\beta-1}$.

6 Exponential decay rates for the reduced system

According to the general scheme proposed in Section 2, we need to check condition **(F1)**, i.e., prove that the induced system $(F, M, \hat{\mu})$ enjoys exponential decay of correlations. Here we use a simplified method to prove exponential decay of correlations for our reduced billiard map. It is mainly based on recent results in [24, 6, 12].

Since billiards have singularities, if the orbit of x approaches the singularity set \mathcal{S}_1 too fast under F , then x may not have a stable/unstable manifold. The situation is kind of complicated here as we have accumulated sequences of new types of singular curves. Indeed we will first show that a small neighborhood of the singular set has small measure.

Lemma 6. *For any $\delta > 0$, the δ -neighborhood of $\mathcal{S}_{\pm 1}$ has measure:*

$$(6.1) \quad \mu(B_\delta(\mathcal{S}_{\pm 1})) \leq C \delta^{\frac{\beta(2\beta-1)}{3\beta^2-3\beta+1}}.$$

Here $B_\delta(\mathcal{S}_{\pm 1}) = \{x \in M : d_M(x, \mathcal{S}_{\pm 1}) \leq \delta\}$ for any $\delta > 0$, and $C > 0$ is a constant, where $d_M(\cdot, \cdot)$ is the distance in M .

Proof. For any given small $\delta > 0$, we first need to find the smallest N_δ such that $\cup_{n \geq N_\delta} M_n \subset \mu_M(B_\delta(\mathcal{S}_1))$. According to Lemma 5, the width of M_N is approximately $\mathcal{O}(N^{-1 - \frac{\beta}{2\beta-1} - \frac{\beta}{(2\beta-1)(\beta-1)}})$. Thus we get $N_\delta = \delta^{-\frac{1}{1 + \frac{\beta}{2\beta-1} + \frac{\beta}{(2\beta-1)(\beta-1)}}}$. This implies that

$$\mu_M(\cup_{n \geq N_\delta} M_n) \sim \delta^{\frac{\beta(2\beta-1)}{3\beta^2-3\beta+1}}.$$

Thus we have

$$\mu(B_\delta(\mathcal{S}_{\pm 1})) \leq \mu(\cup_{n \geq N_\delta} M_n) + C \sum_{n=1}^{N_\delta} \delta n^{-\frac{\beta}{2\beta-1}} \sim \delta^{\frac{\beta(2\beta-1)}{3\beta^2-3\beta+1}}.$$

□

The above lemma implies that almost every point in M has a regular stable (resp. unstable) manifold and there are plenty of reasonable long stable (resp. unstable) manifolds for F . These manifolds are denoted as $\mathcal{W}^{s/u}$. If the forward image of $x \in W \in \mathcal{W}^u$ is almost tangential, then the expansion factors along W may be highly nonuniform. To overcome this difficulty we divide \mathcal{M} into horizontal strips as introduced in [3, 4]. More precisely, one divides \mathcal{M} into countably many sections (called *homogeneity strips*) defined by

$$\mathbf{H}_k = \{(r, \varphi) \in \mathcal{M} : \pi/2 - k^{-2} < \varphi < \pi/2 - (k+1)^{-2}\},$$

and

$$\mathbf{H}_{-k} = \{(r, \varphi) \in \mathcal{M} : -\pi/2 + (k+1)^{-2} < \varphi < -\pi/2 + k^{-2}\},$$

for all $k \geq k_0$ and

$$(6.2) \quad \mathbf{H}_0 = \{(r, \varphi) \in \mathcal{M} : -\pi/2 + k_0^{-2} < \varphi < \pi/2 - k_0^{-2}\},$$

where $k_0 \geq 1$ is a fixed (and usually large) constant, whose value will be chosen to guarantee the one-step expansion – for details, see the end of the proof of Lemma 5.56 in [10] for such a choice of k_0 . We now add the boundary of these homogeneous strips in the the singular set $\mathcal{S}_{\pm 1}^H := \mathcal{S}_{\pm 1} \cup \{\partial \mathbf{H}_k \cup F^{\pm 1} \mathbf{H}_k, k \geq k_0\}$. We denote the resulting collection of stable/unstable manifolds as $\mathcal{W}_H^{s/u}$, and call them the homogeneous invariant manifolds.

Lemma 7. *For the induced map (F, M) , the invariant manifolds in $\mathcal{W}_H^{s/u}$ have the regularity properties: bounded curvature, distortion bounds and absolute continuity.*

The proof of this lemma follows the arguments in [9] Chapter 5 as well as [6], as the induced map (F, M) is essentially the dispersing billiards (with corner points). Thus we will not repeat here.

Lemma 8. (*One-step expansion estimate*) Assume $\beta \in (2, \infty)$. Let W be a short unstable curve in M and $\{W_i\}$ be the collection of smooth components in W . Then

$$(6.3) \quad \liminf_{\delta_0 \rightarrow 0} \sup_{W: |W| < \delta_0} \sum_{i \geq 1} \frac{1}{\Lambda_i} < 1,$$

where the supremum is taken over unstable curves $W \subset M$ and $\Lambda_i = \frac{|FW_i|}{|W_i|}$, $i \geq 1$, denote the minimal local expansion factors of the connected component W_i under the map F .

Proof. Let $W \subset M$ be an unstable curve. Note that the upper bound of (6.3) is only achieved when W intersects one of the accumulating sequences of the singular set.

We consider the worst case by assuming W touches the singular point x_D , and intersects M_N , for $N \geq n_0$, where n_0 depends on the length of W . Note that W crosses s_{N+1} , the boundary of cell M_N , for some $N \geq n_0$. Since s_{N+1} consists of points whose last iteration in the corner series is tangential to the boundary of the table, then it must belong to a cell $\mathcal{F}^{-N}\mathbf{H}_k$, for some $k \geq k_N$. Moreover, if $x \in s_{N+1}$ is very close to the curve s_0 , then its first collision must cross some region $\mathcal{F}^{-1}\mathbf{H}_m$, for some $m \geq m_N$. Define $W_{N,m,k} = W \cap \mathcal{F}^{-N}\mathbf{H}_k \cap \mathcal{F}^{-1}\mathbf{H}_m$. Let $x \in W_{N,m,k}$, and $x_n = \mathcal{F}^n x$, for $n \geq 1$, then Proposition 4 implies that the expansion factor satisfies:

$$\begin{aligned} \Lambda_{N,m,k} &:= \frac{|FW_{N,m,k}|}{|W_{N,m,k}|} \sim N^{1+\frac{\beta}{(2\beta-1)(\beta-1)}} \left(1 + \frac{N^{-\frac{\beta}{2\beta-1}}}{\cos \varphi_1}\right) \cdot \left(1 + \frac{N^{-\frac{\beta(\beta-2)}{(2\beta-1)(\beta-1)}}}{\cos \varphi_N}\right) \\ &\sim N^{1+\frac{\beta}{(2\beta-1)(\beta-1)}} \left(1 + m^2 N^{-\frac{\beta}{2\beta-1}}\right) \cdot \left(1 + k^2 N^{-\frac{\beta(\beta-2)}{(2\beta-1)(\beta-1)}}\right). \end{aligned}$$

Next we will find m_N , which is the smallest integer such that $W \cap \mathcal{F}^{-1}\mathbf{H}_m$ is not empty. Note that the expansion factor for $\mathcal{F}W$ is approximately 1, thus $\mathcal{F}M'_N$ is a cell bounded by $\varphi = \pi$, with r -dimension $\sim N^{-1}$ and φ -dimension $\sim N^{-1}$. Thus it intersects infinitely many homogeneous strips \mathbf{H}_m , with $m \geq m_N$. Thus $k_N \sim N$, which implies that $m_N \sim N$. Further images $\mathcal{F}^i M'_N$ moves away from $\varphi = \pi$, they only intersects homogeneous strips \mathbf{H}_m , with $m < m_N$, for $i = 2, \dots, N_1$. By the symmetric property of the billiard table, when i approaches N , similar patterns repeat, with $\mathcal{F}^N M'_N$ intersecting infinitely many \mathbf{H}_k , for $k \geq m_N$. Thus we have

$$\begin{aligned} \sum_{N \geq n_0} \sum_{m \geq m_N} \sum_{k \geq k_N} \frac{1}{\Lambda_{N,m,k}} &\leq C \sum_{N \geq n_0} \sum_{m \geq m_N} \sum_{k \geq k_N} \frac{N^{-1-\frac{\beta}{(2\beta-1)(\beta-1)}}}{\left(1 + m^2 N^{-\frac{\beta}{2\beta-1}}\right) \cdot \left(1 + k^2 N^{-\frac{\beta(\beta-2)}{(2\beta-1)(\beta-1)}}\right)} \\ &\leq C n_0^{-\frac{2\beta}{2\beta-1}}. \end{aligned}$$

Note that we have assumed W intersects M_N , for all $N \geq n_0$, thus by (5.3),

$$|W| \sim \sum_{N \geq n_0} N^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}} \sim n_0^{-\frac{\beta^2}{(2\beta-1)(\beta-1)}}.$$

Combining the above estimations, we have

$$\sum_{N \geq n_0} \sum_{m \geq N} \sum_{k \geq N} \Lambda_{N,m,k}^{-1} \leq C|W|^{\frac{2(\beta-1)}{\beta}}.$$

Thus by taking $|W|$ small, we can make the above sum < 1 . \square

Given an unstable curve W , a point $x \in W$ and an integer $n \geq 0$, we denote by $r_n(x)$ the distance between $F^n x$ and the boundary of the homogeneous component of $F^n W$ containing $F^n x$. Clearly $r_n(x)$ is a function on W that characterizes the size of the smooth components of $F^n W$. We first state the Growth Lemma, proved in [15], which is key in the analysis hyperbolic systems with singularities. It expresses the fact that the expansion of unstable curves dominates the cutting by singular curves, in a uniform fashion for all sequences. The reason behind this fact is that unstable curves expand at a uniform exponential rate, whereas the cuts accumulate at only a finite number of singular points. The following Growth Lemma can be derived directly from Lemma 8 – the One-step Expansion Estimates, see [9], [15] for details.

Lemma 9. (*Growth Lemma*). *There exist uniform constants $C_{\mathbf{g}}, c > 0$ and $\vartheta \in (0, 1)$, $q = \frac{\beta(2\beta-1)}{3\beta^2-3\beta+1}$, such that, for any probability measure ν supported on an unstable curves W with positive density $d\nu/dm_W \in \mathcal{H}_\gamma$ for some $\gamma \in (0, 1)$, and $n \geq 1$:*

$$(6.4) \quad F^n \nu(r_n < \varepsilon) \leq C_{\mathbf{g}} \vartheta^n \nu(r_n < \varepsilon)^q + c\varepsilon,$$

where m_W is the Lebesgue measure on W .

In [6, 12, 15], the following lemma was proved.

Lemma 10. *If the induced billiard map F satisfies (6.3), and the unstable manifolds have regularities: bounded curvature, distortion bounds and absolute continuity, then there is a hyperbolic horseshoe $\Delta_0 \subset M$ such that*

$$(6.5) \quad \mu_M(x \in M : R(x; F, \Delta_0) > m) \leq C\theta^m \quad \forall m \in \mathbb{N},$$

for some $\theta < 1$, where $R(x; F, \Delta_0)$ is the return time of x to Δ_0 under the map F . Thus the map $F : M \rightarrow M$ enjoys exponential decay of correlations. More precisely, for every pair of dynamically Hölder continuous functions $f, g \in \mathcal{H}_\gamma$ and $n \geq 1$,

$$(6.6) \quad \mathcal{C}_n(f, g, F, \mu_M) \leq C\|f\|_{C^\gamma}\|g\|_{C^\gamma}\theta^n,$$

where $C > 0$ is a uniform constant.

The above results can be extended to variables made at multiple times. Let $f_0, f_1, \dots, f_k \in \mathcal{H}_\gamma$, and $\|f_i\|_\infty = \|f\|_\infty$, $i = 1, \dots, k$. Consider the product $\tilde{f} = f_0 \cdot (f_1 \circ F) \cdots (f_k \circ F^k)$. Furthermore, let $g_0, g_1, \dots, g_k \in \mathcal{H}_\gamma$, and $\|g_i\|_\infty = \|g\|_\infty$, $i = 1, \dots, k$. Consider the product $\tilde{g} = g_0 \cdot (g_1 \circ F) \cdots (g_k \circ F^k)$. Then we can estimate the correlations between observables \tilde{f} and \tilde{g} .

Theorem 11. *There exists $C > 0$, such that for all $n \geq 0$,*

$$\mathcal{C}_n(\tilde{f}, \tilde{g}, F, \mu_M) \leq C \|\tilde{f}\|_{C^\gamma} \|\tilde{g}\|_{C^\gamma} \theta^n,$$

where θ is the same as in (6.6).

Hence we conclude that for $\beta \in (2, \infty)$, the return map $F: M \rightarrow M$ has exponential mixing rates by the above Theorem, thus both condition **(F1)** and **(F2)** are verified. The following lemma was proved in [12].

Lemma 12. *For systems under assumptions **(F1-F2)**, and for the billiard map $\mathcal{F}: \mathcal{M} \rightarrow \mathcal{M}$ and any piecewise Hölder continuous functions $f, g \in \mathcal{H}(\gamma)$ on \mathcal{M} , the correlations (1.2) decay as*

$$(6.7) \quad |\mathcal{C}_n(f, g, \mathcal{F}, \mu)| \leq C \|f\|_{C^\gamma} \|g\|_{C^\gamma} n^{-a} (\ln n)^{1+a},$$

for some constant $C > 0$.

Now we see that Theorem 1 should follow from **(F1)** - **(F2)** for $a = \frac{1}{\beta-1}$, except for the extra logarithmic factor. To improve the upper bound for the decay rates, one needs to analyze the statistical properties of the return time function. In [14], the upper bound for decay rates of correlations was improved by dropping the logarithmic factor.

7 Proof of the main Theorems

A general strategy for estimating the correlation function $\mathcal{C}_m(f, g, \mathcal{F}, \mu)$ for systems with weak hyperbolicity was developed in [12, 14].

Note that Proposition 2 also leads to the the following fact about transitions between cells with different indices.

Proposition 13. *There exist positive constants $c_1 < c_2$, such that for any $n \geq 1$, if $M_m \cap FM_n \neq \emptyset$ then*

$$(7.1) \quad c_1 \sqrt[\beta]{n^{\beta-1}} \leq m \leq c_2 \sqrt[\beta-1]{n^\beta}.$$

Moreover, for any $m \in [c_1 n^{\frac{\beta-1}{\beta}}, c_2 n^{\frac{\beta}{\beta-1}}]$, the transition probability satisfies

$$\mu(M_m | F(M_n)) \sim m^{-1 - \frac{\beta^2}{(\beta-1)(2\beta-1)}} n^{\frac{\beta}{2\beta-1}}.$$

Proof. Without loss of generality we only consider the singular curves near x_D . In particular note that the curves in \mathcal{S}_1 and \mathcal{S}_{-1} are symmetric about r_D in the vicinity of x_D . Let $W \subset M_n$ be a short unstable curve that stretches completely in M_n between two long sides. Then $|FW| \sim n^{-\frac{\beta}{2\beta-1}}$. On the other hand, note that by (5.3), the width of cell M_n is of order $\sim n^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}}$. We assume n_1, n_2 are the two extreme indices, such that

$$n_1 = \min\{m \geq 1 : FM_n \cap M_m \neq \emptyset\}, \quad n_2 = \max\{m \geq 1 : FM_n \cap M_m \neq \emptyset\}.$$

Then

$$\begin{aligned} |FW| &= \sum_{m \geq n_1} |FW \cap M_m| \\ &\sim \sum_{m \geq n_1} m^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}} = n_1^{-\frac{\beta^2}{(2\beta-1)(\beta-1)}}. \end{aligned}$$

Now using the fact that $|FW| \sim n^{-\frac{\beta}{2\beta-1}}$, we can solve for $n_1 \sim n^{\frac{\beta-1}{\beta}}$.

By time - reversibility, one can verify that for the largest index n_2 , if $Fx \in M_{n_2}$, then n is the minimal index, such that $F^{-1}M_{n_2} \cap M_n$ is not empty. Thus above estimation implies $n \sim n_2^{\frac{\beta-1}{\beta}}$, which is equivalent to $n_2 \sim n^{\frac{\beta}{\beta-1}}$.

Next we calculate the transition probability from M_n to M_m , for $m \in [n_1, n]$. Note that $FM_n \cap M_m$ can be approximated by a rectangle with dimensions given by the width of FM_n and M_m . More precisely, $FM_n \cap M_m$ can be approximated as a rectangle with “width” (its r -dimension) $\sim n^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}}$, “height” (the φ -dimension) $\sim m^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}}$ and density weight $\mathcal{O}(1)$. This implies that

$$\mu(M_m | FM_n) \sim \frac{n^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}} m^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}}}{\mu(M_n)} \sim m^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}} n^{\frac{\beta}{2\beta-1}}.$$

□

Although it follows from the above lemma that some points in M_n are mapped to cells with higher indices, one can show that most of the points in M_n indeed have images which belong to cells with much smaller indices.

Lemma 14. *For any small $e > 0$, we define $D_n(e) = \bigcup_{m \geq n^{\frac{\beta-1}{\beta}+e}} M_m$. Then for any n sufficiently large,*

$$\mu(D_n(e) | FM_n) \sim n^{-\frac{e\beta^2}{(2\beta-1)(\beta-1)}} \mu(M_n).$$

In addition, let $\beta_0 = \frac{3+\sqrt{5}}{2}$, then $\mathbb{E}(R \circ |M_n)(x) = e_n \mathbf{I}_{M_n}(x)$, for some $e_n > 0$, with $e_n \leq Cn^{1-\gamma}$, where $C > 0$ is a constant and $\gamma = \frac{1}{(\beta-1)^2}$ for $\beta > \beta_0$ and $\gamma = \frac{1}{\beta}$ for $\beta \in (2, \beta_0]$. And $\gamma = \frac{1}{\beta} + \varepsilon_0$ for $\beta = \beta_0$, where $\varepsilon_0 \in (0, (\frac{1}{\beta-1} - \frac{1}{\beta})/100)$.

Proof. For any n large, we denote $\beta_n := \{m \in [c_1 n^{\frac{\beta-1}{\beta}}, c_2 n^{\frac{\beta}{\beta-1}}]\}$ as the index set of m , such that $M_m \cap FM_n$ is not empty. Combining with Proposition 13, we know for any $m \in \beta_n$,

$$\mu(M_m|FM_n) \sim m^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}} n^{\frac{\beta}{2\beta-1}}.$$

Thus the conditional expectation of $R \circ F$ on M_n satisfies

$$\mathbb{E}(R \circ F|M_n) \sim \sum_{m \in \beta_n} m \cdot \mu(M_m|FM_n) \cdot \mathbf{I}_{M_n} \sim \sum_{m \in \beta_n} m \cdot m^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}} \cdot n^{\frac{\beta}{2\beta-1}} \cdot \mathbf{I}_{M_n}.$$

We let $\beta_0 = \frac{3+\sqrt{5}}{2}$. Note that for $\beta = \beta_0$, the r -dimension of the cell M_n is $\sim n^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}} = n^{-2}$. Thus for $\beta \in (2, \beta_0)$,

$$\mathbb{E}(R \circ F|M_n) \sim R^{1-\frac{1}{\beta}} \mathbf{I}_{M_n};$$

while for $\beta = \beta_0$,

$$\mathbb{E}(R \circ F|M_n) \sim R^{1-\frac{1}{\beta}} \ln R \cdot \mathbf{I}_{M_n} = R^{1-\frac{1}{(\beta-1)^2}} \ln R \cdot \mathbf{I}_{M_n};$$

and for $\beta > \beta_0$,

$$\mathbb{E}(R \circ F|M_n) \sim R^{1-\frac{1}{(\beta-1)^2}} \cdot \mathbf{I}_{M_n}.$$

We take $\gamma = \frac{1}{(\beta-1)^2}$ for $\beta > \beta_0$; $\gamma = \frac{1}{\beta} + \varepsilon_0$ for $\beta = \beta_0$; and $\gamma = \frac{1}{\beta}$ for $\beta < \beta_0$. Here $\varepsilon_0 \in (0, (\frac{1}{\beta-1} - \frac{1}{\beta})/100)$. Thus we have shown that $\mathbb{E}(R \circ F|M_n) = e_n \mathbf{I}_{M_n}$, with $e_n = \mathcal{O}(n^{1-\gamma})$, and $e_n > 0$.

For any small $e > 0$, we define $D_n(e) = \cup_{m \geq n^{\frac{\beta-1}{\beta}+e}} M_m$. Then we have

$$\mu(D_n(e)|FM_n) = \sum_{k \geq n^{\frac{\beta-1}{\beta}+e}} \mu(M_k|FM_n) \sim \sum_{k \geq n^{\frac{\beta-1}{\beta}+e}} k^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}} \cdot n^{\frac{\beta}{2\beta-1}} = n^{-\frac{e\beta^2}{(2\beta-1)(\beta-1)}}.$$

□

This lemma also implies, essentially, that a typical trajectory stays away from those long corner series most of the time. But as β goes to ∞ , more points in FM_n tend to enter long corner series more frequently.

Lemma 15. *For sufficiently large m , any $b \geq 1$, there exists $E_m \subset M_m$, with $\mu(M_m \setminus E_m) \leq m^{-\frac{\beta^2}{2(\beta-1)(3\beta^2-3\beta+1)}} \mu(M_m)$, and any $x \in E_m$, $Fx, F^2x, \dots, F^{b \ln m} x$ all belong to cells with index less than $m^{1-\frac{1}{2\beta}}$.*

Proof. For any $e > 0$ small, it follows from Lemma 14,

$$\mu(D_m(e)|FM_m) = \sum_{n=m^{\frac{\beta-1}{\beta}+e}}^{\infty} \mu(M_n|FM_m) = \mathcal{O}(m^{-\frac{e\beta^2}{(2\beta-1)(\beta-1)}}).$$

Below we choose $e = \frac{1}{2\beta}$, and denote $\mathcal{D}_m := D_m(\frac{1}{2\beta})$. Thus we can neglect points $x \in M_m$ such that $F(x) \in M_m$ with $n > m^{1-\varepsilon} = m^{1-\frac{1}{2\beta}}$, for $\varepsilon = \frac{1}{\beta} - e = \frac{1}{2\beta}$. It remains to estimate the probability that points $y \in M_m$ will come up to M_i , for $i \geq m^{1-\frac{1}{2\beta}}$, within $\mathcal{O}(\ln m)$ iterations of F .

Note that each cell M_m has dimension $\sim m^{-\frac{\beta}{2\beta-1}}$ in the stable direction, dimension $\sim m^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}}$ in the unstable direction, and measure $\mu(M_m) \sim m^{-2-\frac{1}{\beta-1}}$. We first foliate M_m with unstable curves $W_\alpha \subset M_m$ (where α runs through an index set \mathcal{A}). These curves have length $|W_\alpha| \sim m^{-1-\frac{\beta^2}{(2\beta-1)(\beta-1)}}$. Let $\nu_m := \frac{1}{\mu(M_m)}\mu|_{M_m}$ be the conditional measure of μ restricted on M_m . Let $\mathcal{W}_m = \cup_{\alpha \in \mathcal{A}} W_\alpha$ be the collection of all unstable curves, which foliate the cell M_m . Then we can disintegrate the measure ν_m along the leaves W_α . More precisely, for any measurable set $A \subset M_m$,

$$\nu_m(A) = \int_{\mathcal{A}} \nu_\alpha(W_\alpha \cap A) d\lambda(\alpha),$$

where λ is the probability factor measure on \mathcal{A} . For each unstable curve $W_\alpha \in \mathcal{W}$, if $F^m W_\alpha$ crosses \mathcal{D}_m , then $F^l W_\alpha$ is cut into pieces by the boundary of cells in \mathcal{D}_m . Moreover, the largest length of these pieces is $\sim m^{-(1+\frac{\beta^2}{(2\beta-1)(\beta-1)})(1-\frac{1}{2\beta})} = m^{-\frac{3\beta^2-3\beta+1}{2\beta(\beta-1)}}$. According to the growth lemma 9, there exists $\vartheta \in (0, 1)$, such that we have

$$(7.2) \quad F^l \nu_m(\mathcal{D}_m) \leq C_{\mathbf{g}} \vartheta^l F^l \nu_m(\mathcal{D}_m)^{\frac{\beta(2\beta-1)}{3\beta^2-3\beta+1}} + cm^{-\frac{3\beta^2-3\beta+1}{2\beta(\beta-1)}}.$$

Moreover Lemma 14 implies that

$$F \nu_m(\mathcal{D}_m) := \mu(R(F(x)) > m^{1-\frac{1}{2\beta}} | R(x) = m) \leq Cm^{-\frac{\beta}{2(2\beta-1)(\beta-1)}},$$

for some uniform constant $C > 0$.

Now we apply (7.2) to get for any $l = 1, \dots, b \ln m$,

$$\begin{aligned} \nu_m(R(F^l(x)) > m^{1-\frac{1}{2\beta}}) &= F^l \nu_m(\mathcal{D}_m) \\ &\leq C_{\mathbf{g}} \vartheta^l F \nu_m(\mathcal{D}_m)^{\frac{\beta(2\beta-1)}{3\beta^2-3\beta+1}} + cm^{-\frac{3\beta^2-3\beta+1}{2\beta(\beta-1)}} \\ &\leq C \vartheta^l m^{-\frac{\beta^2}{2(\beta-1)(3\beta^2-3\beta+1)}} + cm^{-\frac{3\beta^2-3\beta+1}{2\beta(\beta-1)}}. \end{aligned}$$

Thus we have

$$\sum_{l=1}^{b \ln m} \nu_m(R(F^l(x)) > m^{1-\frac{1}{2\beta}}) \leq C_1 m^{-\frac{\beta^2}{2(\beta-1)(3\beta^2-3\beta+1)}}.$$

This also implies that for any large m , there exists $E_m \subset (R = m)$, with

$$\mu((R = m) \setminus E_m) \leq m^{-\frac{\beta^2}{2(\beta-1)(3\beta^2-3\beta+1)}} \mu(R = m)$$

and any $x \in E_m$, $Fx, F^2x, \dots, F^{b \ln m}x$ all belong to cells with index less than $m^{1-\frac{1}{2\beta}}$. \square

Now we are ready to prove Theorem 1.

The tower in M can be easily and naturally extended to \mathcal{M} , thus we get a the Young's tower with the same base $\Delta_0 \subset M$; and a.e. point $x \in \mathcal{M}$ again properly returns to Δ_0 under \mathcal{F} infinitely many times. Consider the return times to M under \mathcal{F} for $x \in \mathcal{M}$. According to Lemma 5,

$$(7.3) \quad \mu(x \in \mathcal{M} : R(x) > n) \sim \frac{1}{n^{\frac{1}{\beta-1}}}, \quad \forall n \geq 1.$$

For every $m \geq 1$ and $x \in \mathcal{M}$ denote

$$r(x; m, M) = \#\{1 \leq i \leq m : \mathcal{F}^i(x) \in M\}.$$

Let

$$A_m = \{x \in \mathcal{M} : R(x; \mathcal{F}, \Delta_0) > m\},$$

$$B_{m,b} = \{x \in \mathcal{M} : r(x; m, M) > b \ln m\},$$

where $b > 0$ is a constant to be chosen shortly.

By (6.5), we know that

$$\mu(A_m \cap B_{m,b}) \leq C \cdot m \theta^{b \ln m}.$$

Choosing $b = -\frac{2}{\ln \theta}$, then

$$(7.4) \quad \text{const} \cdot m \theta^{b \ln m} \leq \text{const} \cdot m \theta^{-\frac{2}{\ln \theta} \ln m} = \text{const} \cdot m^{-1}.$$

The set $A_m \setminus B_{m,b}$ consists of points $x \in \mathcal{M}$ whose images under m iterations of the map \mathcal{F} return to M at most $b \ln m$ times but never return to the 'base' Δ_0 of Young's tower. Our goal is to show that $\mu(A_m \setminus B_{m,b}) = \mathcal{O}(m^{\frac{1}{1-\beta}})$.

Let $I = [n_0, n_1]$ be the longest interval, within $[1, m]$, between successive returns to M . Without loss of generality, we assume that $m - n_1 \geq n_0$, i.e. the leftover interval to the

right of I is at least as long as the one to the left of it (because the time reversibility of the billiard dynamics allows us to turn time backwards). Due to Lemma 15, for a large portion of typical points $y \in M_{|I|}$ we have $F^t(y) \in M_{m_t}$, where m_t decreases exponentially fast. So there exists $c > 0$, such that

$$m/2 \leq |I| + b|I|^{1-\frac{1}{2\beta}} \ln |I| \leq c|I|,$$

which gives $|I| \geq \frac{m}{2c}$.

Let $G_m = \{x \in A_m \setminus B_{m,b} : |I| \geq \frac{m}{2c}\}$. Thus it is enough to estimate the size of G_m . Since for any $x \in G_m$, one of its forward images belongs to $m_{|I|}$ with $|I| \geq \frac{m}{2c}$. Applying the bound Lemma 5 to the interval I gives

$$(7.5) \quad \mu(G_m) \leq C m \cdot m \cdot m^{-2-\frac{1}{\beta-1}} = C m^{\frac{1}{1-\beta}},$$

(the extra factors of m must be included because the interval I may appear anywhere within the longer interval $[1, m]$, and the measure μ is invariant).

In terms of Young's tower Δ , we obtain

$$(7.6) \quad \mu(x \in \Delta : R(x; \mathcal{F}, \Delta_0) > m) \leq C m^{\frac{1}{1-\beta}}, \quad \forall m \geq 1.$$

This completes the proof of the theorem 1.

8 General models with cusps

In this section, we extend the above results to more general billiard models with cusps at flat points. We mainly consider dispersing billiards with one cusp at a flat point.

The first model is a Lorentz gas with finite horizon, see Figure ?? Let \mathbb{T}^2 be a unit torus, and \mathbf{B}_i , $i = 1, \dots, l_0$ be a finite number of convex scatterers in \mathbb{T}^2 , for some $l_0 > 2$. We assume:

- (1) There exists $K_{\min} > 0$, such that for any point $p \in \partial \mathbf{B}_i$, the curvature $\mathcal{K}(p) \geq K_{\min}$, for $i = 3, \dots, l_0$.
- (2) \mathbf{B}_1 and \mathbf{B}_2 are tangent at a unique flat point P , and the cusp at P satisfies assumption **(h1)**-(**h2**). Moreover, we assume the tangent line of $\partial \mathbf{B}_1$ is perpendicular to the scatter \mathbf{B}_3 .

We define the billiard table as $Q_2 = \mathbb{T}^2 \setminus (\cup_i \mathbf{B}_i)$. Let \mathcal{F}_2 be the billiard map on the collision space \mathcal{M}_2 associated with Q_2 . We define the induced space M_2 as in (2.1), the return time function $R_2 : M_2 \rightarrow \mathbb{N}$, and the induced map $F_2 : M_2 \rightarrow M_2$, such that for any $x \in M_2$, $F_2 x = \mathcal{F}_2^{R_2(x)} x$.

By taking K_0 large, we can check that $\{R_2 = n, n > K_0\} = \{M_n, n > K_0\}$ are essentially identical for both systems $(\mathcal{F}, \mathcal{M})$ and $(\mathcal{F}_2, \mathcal{M}_2)$. The only differences are those level sets with smaller indices, which play very minor roles in the study of decay rates of correlations. Thus one can obtain the same decay rates of correlations.

The second model is the dispersing billiard with corners and a cusp. Let $\Gamma_i, i = 1, \dots, l_0$ be a finite number of convex curves, for some $l_0 > 2$. We assume:

- (1) There exists $K_{\min} > 0$, such that for any point $p \in \Gamma_i$, the curvature $\mathcal{K}(p) \geq K_{\min}$, for $i = 3, \dots, l_0$.
- (2) Γ_1 and Γ_2 are tangent at a unique flat point P , and the cusp at P satisfies assumption **(h1)**-**(h2)**. Moreover, we assume the tangent line of Γ_1 is perpendicular to the boundary Γ_3 .

We define the billiard table as Q_3 , such that $\partial Q_3 = \cup_i \Gamma_i$. Let \mathcal{F}_3 be the billiard map on the collision space \mathcal{M}_3 associated with Q_3 . We define the induced space M_3 as in (2.1), the return time function $R_3 : M_3 \rightarrow \mathbb{N}$, and the induced map $F_3 : M_3 \rightarrow M_3$, such that for any $x \in M_3$, $F_3 x = \mathcal{F}_3^{R_3(x)} x$.

Again by taking K_0 large, we can check that $\{R_2 = n, n > K_0\} = \{R_3 > K_0\}$ are indeed identical for both systems $(\mathcal{F}_2, \mathcal{M}_2)$ and $(\mathcal{F}_3, \mathcal{M}_3)$. The only differences are those level sets with smaller indices, which play very minor roles in the study of decay rates of correlations. Thus one can again obtain the same decay rates of correlations.

9 Appendix: Proof of Proposition 4.

Since the Euclidean norm and the p -norm are uniformly equivalent at the points $x \in M_N$ and Fx , we can use the p -norm in our estimations. Assume $W \subset M_N$ is an unstable curve. Then the expansion factor in the p - metric along $x \in W$ satisfies

$$(9.1) \quad \Lambda(x) = \prod_{n=0}^N (1 + \tau(x_n) \mathcal{B}(x_n)).$$

Let $x_n = \mathcal{F}^n x = (r_n, \varphi_n)$, for $n = 1, \dots, N$, and $\mathcal{K}_n = \mathcal{K}(x_n)$, $\tau_n = \tau(x_n)$. Note that $\mathcal{B}(x_n)$ satisfies the recursive formula

$$(9.2) \quad \mathcal{B}(x_n) = \frac{2\mathcal{K}_n}{\cos \varphi_n} + \frac{1}{\tau_{n-1} + 1/\mathcal{B}(x_{n-1})}.$$

We denote $\lambda_n = \tau_n \mathcal{B}(x_n)$, and use notation $\sin \gamma_n = \cos \varphi_n$, thus

$$(9.3) \quad \lambda_n = \frac{2\tau_n \mathcal{K}(s_n)}{\sin \gamma_n} + \frac{\tau_n}{\tau_{n-1}} \cdot \frac{1}{1 + 1/\lambda_{n-1}},$$

and

$$\Lambda(x) = \prod_{n=0}^N (1 + \lambda_n).$$

By Proposition 2 and (9.3), to estimate the minimal expansion factor $\Lambda(x)$, we first will assume

$$(9.4) \quad \gamma_1 \sim \gamma_N \sim N^{-2/(2+\bar{a})} = N^{-\frac{\beta}{2\beta-1}}.$$

Proposition 16. *For any $x \in M_N$ satisfying (9.4), we have*

- (1) $\lambda_n \sim 1/n$, for $1 \leq n \leq N_1$;
- (2) $\lambda_n \sim 1/N \sim 1/n$, for $N_1 < n < N_3$;
- (3) $\lambda_n \sim 1/(N - n)$, for $N_3 < n < N$.

Item (2) follows directly from Proposition 2(2). This also implies that the total expansion factor during the interval $[N_1, N_3]$ is of order $\mathcal{O}(1)$. Thus we can ignore it when calculation the expansion factors. We will prove two lemmas below corresponding to item (1) and item (3), respectively.

Using (3.32), we know that

$$\frac{2\tau_n \mathcal{K}_n}{\sin \gamma_n} = \frac{D}{n^2} + \mathcal{O}(\alpha_n^3)$$

where $D = \frac{2\bar{a}}{(2+\bar{a})^2}$, and by (3.33),

$$\frac{\tau_{n+1}}{\tau_n} = 1 - \frac{B}{n} + \mathcal{O}(\alpha_n^2),$$

where $B = \frac{4}{2+\bar{a}} = \frac{2\beta}{2\beta-1}$.

Now we use the relation (9.2) to get the estimation for λ_n .

Lemma 17. *For all $1 \leq m < N_1$ we have*

$$(9.5) \quad \lambda_m \geq A \left[m + C_3 \ln m + C_4 \right]^{-1},$$

where $A > 0$ satisfies $A^2 + A(B - 1) = D$, hence $A = \frac{\beta-1}{2\beta-1}$, and $C_3, C_4 > 0$ are sufficiently large constants. Moreover,

$$\left[\prod_{m=0}^{N_1-1} (1 + \tau(X_m) \mathcal{B}(X_m)) \right] > CN^{\frac{\beta-1}{2\beta-1}},$$

for some constant $C > 0$.

Proof. We use induction on m . For $m = 1$ the validity of (9.5) is guaranteed by choosing C_4 large enough. Assume that (9.5) is valid for some $m < N_1$. Due to (9.3) it is enough to verify

$$\frac{D}{\left[m + C'_1 \ln m + C'_2\right]^2} + \frac{A(1 - \frac{B}{m} + \mathcal{O}(\alpha_m^2))}{A + m + C_3 \ln m + C_4} > \frac{A}{m + 1 + C_3 \ln(m + 1) + C_4},$$

provided $C_3, C_4 > 0$ are large enough. Here $B = \frac{4}{2+\bar{a}}$. It is easy to see that

$$\frac{A}{m + 1 + C_3 \ln(m + 1) + C_4} - \frac{A(1 - \frac{B}{m} + \mathcal{O}(\alpha_m^2))}{A + m + C_3 \ln m + C_4} < \frac{A^2 - A(B - 1)}{\Theta},$$

where Θ denotes the product of the two denominators. Thus it is enough to verify

$$\frac{D}{\left[m + C'_1 \ln m + C'_2\right]^2} > \frac{A^2 + A(B - 1)}{\Theta}.$$

We recall that $A^2 + A(B - 1) = D$, since $A = \frac{\bar{a}}{2+\bar{a}}$. Thus it is enough to verify

$$(9.6) \quad \Theta > \left[m + C'_1 \ln m + C'_2\right]^2.$$

The leading term m^2 appears on both sides and cancels out. Keeping only the largest non-cancelling terms on both sides of (9.6) we obtain $C_3 m \ln m > C'_1 m \ln m$, which can be ensured by choosing C_3 large enough. This implies (9.6). Note that due to (9.3), we have

$$\ln \left[\prod_{m=0}^{N_1-1} (1 + \tau(X_m) \mathcal{B}(X_m)) \right] > \sum_{m=1}^{N_1} \left[\frac{A}{m} + \frac{2C_3 \ln m}{m^2} \right],$$

with a sufficiently large constant $C_3 > 0$. Therefore,

$$\ln \left[\prod_{m=0}^{N_1-1} (1 + \tau(X_m) \mathcal{B}(X_m)) \right] > A \ln N_1 + \text{const} > A \ln N + \text{const}.$$

Lastly, note that $A = \frac{\beta-1}{2\beta-1}$, which completes the proof of the lemma. □

Next we consider the expansion factor for $N_3 < n < N$.

Lemma 18. *For any $n \in [N_3 + 1, N - 1]$, we denote $m = N - n + 1$. Then we have*

$$(9.7) \quad \lambda_{N-m} \geq A \left[m + C_3 \ln m + C_4 \right]^{-1},$$

where $A > 0$ satisfies $A^2 - A(B - 1) = D$, with $B = \frac{4}{2+a}$, hence $A = \frac{\beta}{2\beta-1}$, and $C_3, C_4 > 0$ are sufficiently large constants. Moreover,

$$\left[\prod_{n=N_3}^{N-1} (1 + \lambda_n) \right] > CN^{\frac{\beta}{2\beta-1}},$$

for some constant $C > 0$.

Proof. By the time reversibility, we denote $m = N - n + 1$. Assume that (9.7) is valid for some $m < N_1$. Due to (9.3) and the time reversibility, it is enough to verify

$$\frac{D}{\left[m + C'_1 \ln m + C'_2 \right]^2} + \frac{A(1 + \frac{B}{m} + \mathcal{O}(\alpha_{N-m}))}{A + m + C_3 \ln m + C_4} > \frac{A}{m - 1 + C_3 \ln(m - 1) + C_4},$$

provided $C_3, C_4 > 0$ are large enough. Thus one can check that a sufficient condition for the above inequality is $A^2 - A(B - 1) = D$, which implies that $A = \frac{\beta}{2(\beta-1)}$, as we claimed.

Note that we have

$$\ln \left[\prod_{n=N_3}^{N-1} (1 + \lambda_n) \right] > \sum_{m=1}^{N-N_3} \left[\frac{A}{m} + \frac{2C_3 \ln m}{m^2} \right],$$

with a sufficiently large constant $C_3 > 0$. Therefore,

$$\ln \left[\prod_{n=N_3}^{N-1} (1 + \lambda_n) \right] > A \ln(N - N_3) + \text{const} > A \ln N + \text{const}.$$

Lastly, note that $A = \frac{\beta}{2\beta-1}$, which completes the proof of the lemma. \square

After the last collision, the particle leaves the cusp and flies back to the boundary Γ_3 . According to (9.7), during the exit period, $\lambda_n \sim (N - n + 1)^{-1}$, and

$$\tau_n \sim \frac{\alpha_n^{\frac{\beta}{\beta-1}}}{(N - n + 1)\alpha_n} \sim N^{-\frac{\beta}{(2\beta-1)(\beta-1)}} (N - n + 1)^{-\frac{2\beta}{2\beta-1}}.$$

Thus

$$(9.8) \quad \mathcal{B}(x_n) \sim \lambda_n / \tau_n \sim N^{\frac{\beta}{(2\beta-1)(\beta-1)}} (N - n + 1)^{\frac{1}{2\beta-1}}.$$

Although the last collision we have $\tau_N \sim 1$, but (9.8) still holds for $n = N$, as it was derived from previous collisions before and at x_N , which does not depend on τ_N . Thus the expanding factor contributed by the collisions at x_N satisfies

$$1 + \tau_N \mathcal{B}(x_N) \sim N^{\frac{\beta}{(2\beta-1)(\beta-1)}},$$

under assumption (9.4).

On the other hand, for the last collision, when γ_N fails (9.4), we have

$$\begin{aligned} \frac{|D_{x_N} \mathcal{F}(dx_N)|_p}{|dx_N|_p} &= 1 + \tau(x_N) \mathcal{B}(x_N) = 1 + \tau(x_N) \left(\mathcal{B}^-(x_N) + \frac{2\mathcal{K}(s_N)}{\cos \varphi_N} \right) \\ &\sim \tau_{\min} \mathcal{B}^-(x_N) + \tau_{\min} \frac{2\mathcal{K}(s_N)}{\cos \varphi_N} \sim N^{\frac{\beta}{(2\beta-1)(\beta-1)}} + \frac{\alpha_1^{\frac{\beta-2}{\beta-1}}}{\cos \varphi_N} \\ &\sim N^{\frac{\beta}{(2\beta-1)(\beta-1)}} \left(1 + \frac{N^{-\frac{\beta(\beta-3)}{(2\beta-1)(\beta-1)}}}{\cos \varphi_N} \right), \end{aligned}$$

where we have used the fact that $\mathcal{B}^-(x_N) > 0$, and the free path between x_N and Fx is uniformly bounded away from τ_{\min} .

Combining the above facts, we have

$$\Lambda(x) \sim N^{\frac{\beta-1}{2\beta-1}} \cdot N^{\frac{\beta}{2\beta-1}} \cdot N^{\frac{\beta}{(2\beta-1)(\beta-1)}} = N^{1+\frac{\beta}{(2\beta-1)(\beta-1)}}.$$

For points $x \in M_N$, when γ_1 fails to satisfy (9.4), the expression between the first and second collision is

$$\begin{aligned} \frac{|D_{x_1} \mathcal{F}(dx_1)|_p}{|dx_1|_p} &= 1 + \tau(x_1) \mathcal{B}(x_1) = 1 + \tau(x_1) \left(\mathcal{B}^-(x_1) + \frac{2\mathcal{K}(s_1)}{\cos \varphi_1} \right) \\ &= 1 + \tau(x_1) \left(\frac{2\mathcal{K}(s_1)}{\cos \varphi_1} + \frac{1}{\tau(x) + \frac{1}{\mathcal{B}^-(x) + \frac{2\mathcal{K}(r)}{\cos \varphi}}} \right) \\ &\geq 1 + \tau(x_1) \left(\frac{2\mathcal{K}(s_1)}{\cos \varphi_1} + \frac{1}{\tau_{\max} + \frac{1}{2\mathcal{K}_{\min}}} \right) \\ &= 1 + \frac{2\tau(x_1)\mathcal{K}(s_1)}{\cos \varphi_1} + \mathcal{O}(\tau(x_1)) \sim 1 + \frac{N^{-\frac{\beta}{2\beta-1}}}{\cos \varphi_1}, \end{aligned}$$

where we have used the fact that $\mathcal{B}^-(x) > 0$ for any unstable vector in $\mathcal{C}^u(x)$, and our estimations on α_1 and γ_1 in Proposition 2.

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